

\mathbb{R}/\mathbb{Z} -VALUED INDEX THEORY VIA GEOMETRIC K -HOMOLOGY

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ABSTRACT. A model of K -homology with coefficients in a mapping cone using the framework of the geometric cycles of Baum and Douglas is developed. In particular, this leads to a geometric realization of K -homology with coefficients in \mathbb{R}/\mathbb{Z} . In turn, this group is related to the relative η -invariant via index pairings.

1. INTRODUCTION

The goal of this paper is the construction of a relative group in geometric K -homology. This construction is inspired by ideas of Baum and Douglas (see [5]), Karoubi (see [21, Section 2.13]) and Stong (see [27, Chapter 1]). An application is the construction of geometric models for K -homology with coefficients; of particular interest is the coefficient group \mathbb{R}/\mathbb{Z} . The reader is directed to [20] (in particular, Section 6) for a discussion on the relationship between geometric K -homology and \mathbb{R}/\mathbb{Z} -valued index theory. The reader should also note that the starting point for the construction considered here was [20, Remark 6.12].

To motivate our construction, we briefly discuss various pairings between K -theory and K -homology. Let X denote a finite CW-complex, $K^*(X)$ its K -theory and $K_*(X)$ its K -homology. The index pairing between K -theory and K -homology is defined as a map $K^p(X) \times K_p(X) \rightarrow \mathbb{Z}$. It is useful to have an explicit realization of this pairing depending on the specific choice of cocycles used to model K -theory (e.g., vector bundles, projective modules or projections) and the specific choice of cycles in K -homology (e.g., Baum-Douglas cycles, Kasparov cycles, or Extensions). For example, in the context of projections and Kasparov cycles, the reader can find such a formula in [19, Section 8.2]. In fact, this formula (and many similar formulae) factor as follows:

$$(1) \quad K^p(X) \times K_p(X) \rightarrow K_0(pt) \cong \mathbb{Z}$$

where the isomorphism between $K_0(pt)$ and \mathbb{Z} is given (depending on context) either by the topological or analytic index map.

It is natural to ask what the situation is for pairings in K -theory and K -homology with coefficients. As such, let $K^*(X; \mathbb{R}/\mathbb{Z})$ denote the K -theory with coefficients in \mathbb{R}/\mathbb{Z} of X . There is now an index pairing $K^p(X; \mathbb{R}/\mathbb{Z}) \times K_p(X) \rightarrow \mathbb{R}/\mathbb{Z}$ and (in [22]) Lott showed that it is realized analytically by the relative η -invariant. Needless to say, the relative η -invariant is an important invariant which has been studied extensively since being introduced by Atiyah, Patodi, and Singer in [1, 2, 3]. Based

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on the discussion in the previous paragraph (in particular, Equation 1), one would like to write this pairing in the following form:

$$(2) \quad K^p(X; \mathbb{R}/\mathbb{Z}) \times K_p(X) \rightarrow K_0(pt; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$$

Thus, to obtain a suitable *geometric* realization of this pairing, we need to understand $K_0(pt; \mathbb{R}/\mathbb{Z})$ geometrically.

Analytically, $K_*(X; \mathbb{R}/\mathbb{Z})$ is given by $KK^*(C(X), SC_\phi)$ where $\phi : \mathbb{C} \hookrightarrow N$, N is a II_1 -factor, and C_ϕ denotes the mapping cone of ϕ . More generally, we let $\phi : B_1 \rightarrow B_2$ be any unital $*$ -homomorphism between (unital) C^* -algebras, B_1 and B_2 . Our goal is the construction of a geometric model for K -homology with coefficients in the mapping cone of ϕ . In other words, a geometric realization of the KK -theory group: $KK(C(X), C_\phi)$ where C_ϕ is the mapping cone of ϕ . The connection between this framework and K -homology with coefficients is discussed in detail in Example 5.3.

Additional notation is required to review the Baum-Douglas model of K -homology and its generalization to K -homology with coefficients in a unital C^* -algebra (i.e., a geometric model for $KK(C(X), B)$ where B is a unital C^* -algebra). Recall that X denotes a finite CW-complex. The C^* -algebra of continuous B -valued functions on X is denoted by $C(X, B)$ and the Grothendieck group of (isomorphism classes of) finitely generated projective Hilbert B -module bundles over X is denoted by $K^0(X; B)$. It is well-known (for example, see [26, Proposition 2.17]) that

$$K^0(X; B) \cong K_0(C(X, B)) \cong K_0(C(X) \otimes B)$$

We will refer to finitely generated projective Hilbert B -modules bundles simply as B -module bundles; all bundles in this paper are assumed to be locally trivial. The reader should recall (see [5, 6, 7, 8, 9]) that a cycle (over X) in the Baum-Douglas model (for K -homology) is given by a triple, (M, E, f) , where M is a smooth compact *spin*^c-manifold, E is a smooth Hermitian vector bundle over M , and $f : M \rightarrow X$ is a continuous map. Addition of cycles is defined using disjoint union. The geometric K -homology group (denote $K_*(X)$) is given by equivalence classes of cycles under the relation generated by three elementary operations: disjoint union/direct sum, bordism and vector bundle modification. The reader can find a nice treatment of the construction of this model in [9].

This model has been generalized to produce models for $KK(C(X), B)$ (recall that B denotes a unital C^* -algebra); one replaces the vector bundle, E , with a (finitely generated projective) Hilbert B -module bundle. The operations, relations, and map to $KK(C(X), B)$ are analogous to those on the original cycles of Baum and Douglas. Moreover, if $\phi : B_1 \rightarrow B_2$ is as above, then there is an induced map

$$\phi_* : KK(C(X), B_1) \rightarrow KK(C(X), B_2)$$

which is defined at the level of geometric cycles via $(M, E_{B_1}, f) \mapsto (M, E_{B_1} \otimes_\phi B_2, f)$. More details on this model can be found in [28].

Analytically (i.e., in the framework of C^* -algebras), the mapping cone of ϕ (denoted C_ϕ) leads to the following six term exact sequence:

$$\begin{array}{ccccc} KK^0(C(X), B_1) & \xrightarrow{\phi_*} & KK^0(C(X), B_2) & \longrightarrow & KK^0(C(X), C_\phi) \\ \uparrow & & & & \downarrow \\ KK^1(C(X), C_\phi) & \longleftarrow & KK^1(C(X), B_2) & \xleftarrow{\phi_*} & KK^1(C(X), B_1) \end{array}$$

Two geometric models for $KK(C(X), C_\phi)$ are developed (see Sections 3 and 4). A key part of both is the construction of an exact sequence of the same form as the one produced by the mapping cone construction (see Theorems 3.12 and 4.19).

The first model is defined using cycles of the form $(M, [(E_{B_1}, F_{B_1}, \varphi)], f)$ where M and f are as in the Baum-Douglas model and $[E_{B_1}, F_{B_1}, \varphi]$ is an element in $K_0(C(M) \otimes C_\phi)$. We denote the associated K -homology group by $\bar{K}_*(X; \phi)$. While these cycles are defined in a natural way, it is unclear (to the author) how to *naturally* incorporate certain constructions from geometric K -homology into this framework - in particular, construction related to the bordism relation. For example, suppose (M, F_{B_1}, f) is a geometric cycle in $KK(C(X), B_1)$ whose image under ϕ_* is a boundary; that is, $(M, F_{B_1} \otimes_\phi B_2, f) = \partial(W, E_{B_2}, f)$. Based on the exact sequence in Theorem 3.12, we have a cycle $(\hat{M}, [\hat{E}_{B_1}, \hat{F}_{B_1}, \varphi], g)$ such that

$$(M, F_{B_1}, f) \sim (\hat{M}, \hat{E}_{B_1}, g) \dot{\cup} (-\hat{M}, \hat{F}_{B_1}, g)$$

(where “ $-$ ” denotes taking the opposite $spin^c$ -structure). However, it is rather difficult to *explicitly* find such a cycle.

This consideration leads us to a second model which is defined using cycles more closely related to the bordism relation in geometric K -homology. The cycles in this model are given by $(W, (E_{B_2}, F_{B_1}, \alpha), f)$ where W is a smooth compact $spin^c$ -manifold with boundary, E_{B_2} is a Hilbert B_2 -module bundle over W , F_{B_1} is a Hilbert B_1 -module bundle over ∂W , $\alpha : E_{B_2}|_{\partial W} \cong F_{B_1} \otimes_\phi B_2$, and $f : W \rightarrow X$ is a continuous map. We denote the associated K -homology group by $K_*(X; \phi)$. It should be evident to the reader that the problem discussed in the previous paragraph becomes a non-issue for this model. Of course, the two models lead to isomorphic theories; we discuss an explicit isomorphism in Section 5.

A summary of the content of the paper seems in order. In Section 2, we discuss the K -theory data required for each of our models. In Sections 3 and 4, the two geometric models for $KK(C(X), C_\phi)$ are defined. An isomorphism between these two theories is given in Section 5. Finally, in Section 6, we specialize to the case of $\phi : \mathbb{C} \rightarrow N$ (where N is a II_1 -factor). As noted above, this special case gives a geometric model for K -homology with coefficients in \mathbb{R}/\mathbb{Z} .

We have assumed that the reader is familiar with the Baum-Douglas model for K -homology, the basic properties of Hilbert C^* -module bundles, and some basic KK -theory. The notation of [9], [25], and [26] (in particular, see Section 2 of this paper) are all used in this paper. Details on the “straightening the angle” technique can be found in [12] and [25, Appendix]. In Section 6, a construction which is similar to one in [3, Section 5] is discussed. This construction also uses the geometric model for $K_*(X; \mathbb{Z}/k\mathbb{Z})$ discussed in [13] and [14].

The reader will notice that the constructions in this paper are almost completely geometric (an exception is Section 3.2). Since the heart of index theory is the interaction between analysis and geometry, they may wonder if there is an analytic side to the constructions considered here. This is indeed the case; for a detailed development see [15]. Briefly, this work involves two main themes:

- (1) The construction of an explicit (i.e., defined at the level of cycle) isomorphism from $K_*(X; \phi)$ to $KK^{*+1}(C(X), C_\phi)$;
- (2) Connecting the index map $K_0(pt; \mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}$ to higher APS-index theory (see [24] and references therein).

2. SOME REMARKS ON K -THEORY

2.1. K -theory classes for the Karoubi type model. To begin, we recall the construction of a relative K -theory group in [21, Section 2.13]. In our context, this construction leads to a realization of the K -theory of a certain mapping cone (see Proposition 2.2 below). As above, $\phi : B_1 \rightarrow B_2$ is a unital $*$ -homomorphism and X is a finite CW-complex. Let $\Gamma(X; \phi)$ be the set of cocycles, (E, F, φ) , where E and F are B_1 -module bundles over X and $\varphi : E \otimes_{\phi} B_2 \rightarrow F \otimes_{\phi} B_2$ is an isomorphism of B_2 -module bundles. Two cocycles (E, F, φ) and (E', F', φ') are isomorphic if there exists an isomorphism of B_1 -module bundles, $\beta_1 : E \rightarrow E'$ and $\beta_2 : F \rightarrow F'$, which fits into the following commutative diagram:

$$\begin{array}{ccc} E \otimes_{\phi} B_2 & \xrightarrow{\alpha} & F \otimes_{\phi} B_2 \\ \phi^*(\beta_1) \downarrow & & \downarrow \phi^*(\beta_2) \\ E' \otimes_{\phi} B_2 & \xrightarrow{\alpha'} & F' \otimes_{\phi} B_2 \end{array}$$

A cocycle is elementary if $E = F$ and α is homotopic to $Id_{E \otimes_{\phi} B_2}$. We can add cocycles using direct sum.

Definition 2.1. Let $K^0(X; \phi)$ be the quotient of $\Gamma(X; \phi)$ by the equivalence relation $\epsilon \sim \epsilon'$ if there exists elementary cocycles ε and ε' such that $\epsilon + \varepsilon \cong \epsilon' + \varepsilon'$. Also let $K^1(X; \phi) := \tilde{K}^0(SX; \phi)$ where SX is the suspension of X and \tilde{K} denotes reduced K -theory.

The next proposition summarizes the basic properties of $K^*(X; \phi)$ (see [21, Chapter 2] for details).

Proposition 2.2. $K^*(X; \phi)$ is an abelian group which is naturally isomorphic to $K_*(C(X) \otimes C_{\phi})$ where C_{ϕ} is the mapping cone of ϕ . That is,

$$C_{\phi} = \{(f, b_1) \in C([0, 1], B_2) \oplus B_1 \mid f(0) = \phi(b_1)\}$$

In particular, Bott periodicity leads to the following six term exact sequence

$$\begin{array}{ccccc} K_0(C(X) \otimes B_1) & \xrightarrow{\phi_*} & K_0(C(X) \otimes B_2) & \xrightarrow{r} & K^1(X; \phi) \\ \uparrow \partial & & & & \downarrow \partial \\ K^0(X; \phi) & \xleftarrow{r} & K_1(C(X) \otimes B_2) & \xleftarrow{\phi_*} & K_1(C(X) \otimes B_1) \end{array}$$

The maps in this exact sequence are defined explicitly at the level of cocycles. For example, in the case of $K^0(X; \phi)$, if u is a unitary in $M_n(B)$ and (E, F, φ) is a cocycle in $K^0(X; \phi)$, then

$$r([u]) = [(X \times A^n, X \times A^n, u)] \text{ and } \partial[E, F, \varphi] = [E] - [F]$$

Remark 2.3. It follows from this proposition that $K^*(X; \phi)$ has many properties in common with K -theory. For example, this theory has a version of Thom isomorphism, is a module over K -theory, and if M is a compact $spin^c$ -manifold then $K^*(M; \phi)$ satisfies a form of Poincaré duality; namely,

$$K^*(M; \phi) \cong KK^*(\mathbb{C}, C(M) \otimes C_{\phi}) \cong_{PD_M} KK^{*+dim(M)}(C(M), C_{\phi})$$

2.2. K -theory classes for the bordism type model. In this section, we discuss the K -theory construction relevant for the second model for $KK(C(X), C_\phi)$. The results from this section will not be used until Section 4.

Definition 2.4. Let Q be a compact manifold with boundary (its boundary is denoted by ∂Q) and $\phi : B_1 \rightarrow B_2$ be a unital $*$ -homomorphism. Then

$$C^*(Q, \partial Q; \phi) := \{(f, g) \in C(Q, B_2) \oplus C(\partial Q, B_1) \mid f|_{\partial Q} = \phi \circ g\}$$

The reader should note that $C^*(Q, \partial Q; \phi)$ is a C^* -algebra and fits into the following pullback diagram:

$$\begin{array}{ccc} C^*(Q, \partial Q; \phi) & \longrightarrow & C(\partial Q, B_1) \\ \downarrow & & \downarrow \phi_* \\ C(Q, B_2) & \xrightarrow{|\partial} & C(\partial Q, B_2) \end{array}$$

The next two propositions follow from [23, Sections 2 and 3]. In regards to applying the results in [23], the reader should note that the map $C(Q, B_2) \rightarrow C(\partial Q, B_2)$ is onto.

Proposition 2.5. Let Q be a compact manifold with boundary. Then

$$K^0(Q, \partial Q; \phi) := K_0(C^*(Q, \partial Q; \phi)) = \text{Grothendieck group}([E_{B_2}, F_{B_1}, \alpha])$$

where

- (1) E_{B_2} is a finitely generated projective Hilbert B_2 -module bundle over Q ;
- (2) F_{B_1} is a finitely generated projective Hilbert B_1 -module bundle over ∂Q ;
- (3) α is an isomorphism from $E_{B_2}|_{\partial Q}$ to $\phi_*(F_{B_1}) := F_{B_1} \otimes_\phi B_2$.
- (4) $[\cdot]$ denotes taking the isomorphic class.

Proposition 2.6. Let Q be a compact manifold with boundary. Then, the following sequence is exact:

$$\begin{array}{ccccc} K_1(C^*(Q, \partial Q; \phi)) & \longrightarrow & K_1(C(Q, B_2)) \oplus K_1(C(\partial Q, B_1)) & \longrightarrow & K_1(C(\partial Q, B_2)) \\ \uparrow & & & & \downarrow \\ K_0(C(\partial Q, B_2)) & \longleftarrow & K_0(C(Q, B_2)) \oplus K_0(C(\partial Q, B_1)) & \longleftarrow & K_0(C^*(Q, \partial Q; \phi)) \end{array}$$

3. MODEL FOR $KK(C(X), C_\phi)$ VIA RELATIVE K -THEORY

3.1. Cycles and relations. Throughout, X and Y are finite CW-complexes with Y a subcomplex of X . We refer to (X, Y) as a CW-pair.

Definition 3.1. A cycle (over (X, Y) with respect to ϕ) is a triple, $(M, (E, F, \varphi), f)$ where

- (1) M is a smooth compact $spin^c$ -manifold with boundary;
- (2) $[(E, F, \varphi)]$ is a class in $K^0(M; \phi)$;
- (3) $f : M \rightarrow X$ is a continuous map such that $f(\partial M) \subset Y$;

There is a natural definition of isomorphism for such cycles (see page 75 of [28] for details). When we refer to a “cycle”, we will in fact be referring to an isomorphism class of a cycle. We can add (isomorphism classes of) cycles using the disjoint union operation. That is,

$$(M, [(E, F, \varphi)], f) + (M', [(E', F', \varphi')], f') = (M \dot{\cup} M', [(E, F, \varphi)] \dot{\cup} [(E', F', \varphi')], f \dot{\cup} f')$$

Associated to a cycle is its “opposite” which is obtained from the same data with the $spin^c$ -structure on the manifold reversed; we denote M with its opposite $spin^c$ -structure by $-M$.

Definition 3.2. A bordism (over (X, Y) with respect to ϕ) is given by $(W, [(E, F, \varphi)], g, h)$, where

- (1) W is a smooth compact $spin^c$ -manifold with boundary;
- (2) $[(E, F, \varphi)]$ is a class in $K^0(W; \phi)$;
- (3) $g : W \rightarrow X$ is a continuous map;
- (4) $h : \partial W \rightarrow \mathbb{R}$ with regular values ± 1 and $g(h^{-1}[-1, 1]) \subseteq Y$.

From this data, we obtain two cycles:

$$(M_+, [(E, F, \varphi)]|_{M_+}, g|_{M_+}) \text{ and } (M_-, [(E, F, \varphi)]|_{M_-}, g|_{M_-})$$

where $M_+ := h^{-1}([1, \infty))$ and $M_- := h^{-1}((-\infty, -1])$. We then say that the cycles, $(M_+, [(E, F, \varphi)]|_{M_+}, g|_{M_+})$ and $(-M_-, [(E, F, \varphi)]|_{M_-}, g|_{M_-})$, are bordant and denote this by

$$(M_+, [(E, F, \varphi)]|_{M_+}, g|_{M_+}) \sim_{bor} (-M_-, [(E, F, \varphi)]|_{M_-}, g|_{M_-})$$

Proposition 3.3. *Bordism is an equivalence relation on the set of cycles.*

Proof. The proof is very similar to the proof in the case of $K_*(X; A)$ (see [28, Lemma 2.1.10]) and is left to the reader. \square

Definition 3.4. Let $(M, [E, F, \varphi], f)$ be a cycle and V a $spin^c$ -vector bundle of even rank over M . Then the vector bundle modification of $(M, [E, F, \varphi], f)$ by V is defined to be:

$$(M^V, \pi^*([(E, F, \varphi)]) \otimes_{\mathbb{C}} \beta_V, f \circ \pi)$$

where

- (1) $\mathbf{1}$ is the trivial real line bundle over M (i.e., $M \times \mathbb{R}$).
- (2) $M^V = S(V \oplus \mathbf{1})$ (i.e., the sphere bundle of $V \oplus \mathbf{1}$).
- (3) β_V is the “Bott element” in $K^0(M^V)$ (see Section 2.5 of [25] for the construction of this element).
- (4) $\otimes_{\mathbb{C}}$ denotes the $K^0(M^V)$ -module structure of $K^0(M^V; \phi)$.
- (5) $\pi : M^V \rightarrow M$ is the bundle projection.

The vector bundle modification of a cycle $(M, [E, F, \varphi], f)$ by V is often denoted by $(M, [E, F, \varphi], f)^V$.

A cycle, $(M, [E, F, \varphi], f)$, is called even (resp. odd) if the dimensions of the connected components of M are all even (resp. odd) dimensional.

Definition 3.5. Let (X, Y) be a finite CW-pair and \sim be the equivalence relation generated by bordism and vector bundle modification (i.e., $(M, [E, F, \varphi], f)^V \sim (M, [(E, F, \varphi)], f)$). Then

$$\begin{aligned} \bar{K}_0(X, Y; \phi) &:= \{ \text{even cycles} \} / \sim \\ \bar{K}_1(X, Y; \phi) &:= \{ \text{odd cycles} \} / \sim \end{aligned}$$

We denote $\bar{K}_*(X, \emptyset; \phi)$ by $\bar{K}_*(X; \phi)$.

Proposition 3.6. *If (X, Y) is a finite CW-pair, then $\bar{K}_*(X, Y; \phi)$ is a graded abelian group.*

Proof. The disjoint union operation gives $\bar{K}_*(X, Y; \phi)$ the structure of an abelian semi-group. Any cycle which is a boundary (for example, the empty cycle) gives the identity (i.e., zero) class and the opposite of a cycle gives an inverse. \square

The functorial properties of the group $\bar{K}_*(X, Y; \phi)$ are similar to those of K -homology with coefficients in a C^* -algebra. For example, if Z is another finite CW-complex and $g : X \rightarrow Z$ is a continuous map, then $g_*(M, [E, F, \varphi], f) := (M, [E, F, \varphi], g \circ f)$. The boundary map $\partial : K_*(X, Y; \phi) \rightarrow K_*(Y; \phi)$ is defined at the level of cycles via

$$(M, [E, F, \varphi], f) \mapsto (\partial M, [E, F, \varphi]|_{\partial M}, f|_{\partial M})$$

3.2. Isomorphism with analytic theory. To begin, recall that associated to the items in a geometric cycle are the following KK -theory classes:

- (1) $[D_M] \in KK^*(C_0(M), \mathbb{C})$, the class of the Dirac operator on M ;
- (2) $[(E, F, \varphi)] \in KK^0(\mathbb{C}, C_0(M) \otimes C_\phi) \cong K^0(M; \phi)$;
- (3) $[f] \in KK(C_0(X - Y), C_0(M))$;

Combining these classes leads to the following map:

$$(3) \quad \alpha : (M, (E, F, \varphi), f) \mapsto [f] \otimes_{C_0(M)} [(E, F, \varphi)] \otimes_{C_0(M)} \Delta$$

where Δ is the image of the Dirac class under the map on K -homology induced from the diagonal inclusion of M into $M \times M$. This map can also be written as

$$\alpha : (M, [E, F, \varphi], f) \mapsto f_*(PD_M([E, F, \varphi]))$$

where PD_M denotes the Poincare duality map discussed in Remark 2.3.

Proposition 3.7. *The map $\alpha : \bar{K}_*(X, Y; \phi) \rightarrow KK^*(C_0(X - Y), C_\phi)$ (as defined in Equation 3) is well-defined.*

Proof. We begin with the bordism relation. Let $(W, (\tilde{E}, \tilde{F}, \tilde{\varphi}), f)$ be a bordism with boundary $(M, (E, F, \varphi), f)$. It is well known (see for example [9, Theorem 3.5 iii]) that $\partial[D_W] = [D_M]$. In addition, $f = F \circ r$ where $r : M \hookrightarrow W$ is the inclusion of the boundary. Using the fact that $[r] \otimes [\partial] = 0$ and basic properties of KK -theory, we obtain

$$\begin{aligned} \alpha(M, (E, F, \varphi), f) &= [f] \otimes_{C_0(M)} [(E, F, \varphi)] \otimes_{C(M)} [\text{diag}_M] \otimes_{C(M)} [D_M] \\ &= [F \circ r] \otimes_{C(M)} [E, F, \varphi] \otimes_{C(M)} [\text{diag}_M] \otimes [\partial] \otimes [D_W] \\ &= [F] \otimes [r] \otimes [\partial] \otimes [\tilde{E}, \tilde{F}, \tilde{\varphi}] \otimes [\text{diag}_W] \otimes [D_W] \\ &= 0 \end{aligned}$$

Next, the case of vector bundle modification is considered. Let $(M, [E, F, \varphi], f)$ be a cycle and V a spin^c -vector bundle over M with even-dimensional fibers. Let $\pi : M^V \rightarrow M$ denote the projection map and $s : M \rightarrow M^V$ the inclusion of M via the “north pole”. Using standard results on wrong-way maps (see for example [11]), we have that $s! = PD_{M^V}^{-1} s_* PD_M$. All of this, along with the fact that $\pi \circ s = \text{id}_M$, leads to

$$\begin{aligned} \alpha((M, [E, F, \varphi], f)^V) &= \alpha(M^V, s!([E, F, \varphi], f \circ \pi)) \\ &= (f_* \circ \pi_*) PD_{M^V} s!([E, F, \varphi]) \\ &= f_* \circ (\pi_* \circ s_*) PD_M([E, F, \varphi]) \\ &= f_* PD_M([E, F, \varphi]) \\ &= \alpha((M, [E, F, \varphi], f)) \end{aligned}$$

□

Following [11], we introduce conditions which allow for the construction of an inverse to the map α . Namely, suppose that there exists compact $spin^c$ -manifold (possibly with boundary) Z and continuous maps $h : (X, Y) \rightarrow (Z, \partial Z)$ and $g : (Z, \partial Z) \rightarrow (X, Y)$ such that $g \circ h$ is homotopic to the identity map on (X, Y) . (We note that all finite CW-pairs satisfy this condition.) The map from $KK^*(C_0(X - Y), C_\phi) \rightarrow \bar{K}_{*+1}(X, Y; \phi)$ is defined at the level of cycles via

$$\beta : \xi \in KK(C_0(X - Y), SC_\phi) \mapsto (Z, PD_Z^{-1}(h_*(\xi)), g)$$

Lemma 3.8. *Suppose $g : N \rightarrow M$ is an embedding of compact $spin^c$ -manifolds with the codimension of $g(N)$ even-dimensional. Also, let $(M, [E, F, \varphi], f)$ be a cycle. Then*

$$(M, g!(E, F, \varphi), f) \sim (N, (E, F, \varphi), f \circ g)$$

Proof. The proof is essentially the same as the proof of Lemma 2.3.4 in [28]. The details are left to the reader. □

Lemma 3.9. *Let X be a compact Hausdorff space. Then $\alpha \circ \beta = id$. Moreover, if X is a smooth compact $spin^c$ -manifold and we take $Z = X$, $g = h = id$, then $\beta \circ \alpha = id$.*

Proof. The first equality follows from

$$\begin{aligned} \alpha(\beta(\xi)) &= \alpha(Z, PD_M^{-1}(h^*(\xi)), g) \\ &= g_*((PD_M \circ PD_M^{-1} \circ h_*)(\xi)) \\ &= (g_* \circ h_*)(\xi) \\ &= \xi \end{aligned}$$

For the second equality, we have

$$\begin{aligned} \beta(\alpha(M, (E, F, \varphi), f)) &= (X, (PD_X^{-1} \circ f_* \circ PD_X)([E, F, \varphi]), id_X) \\ &= (X, f!(E, F, \varphi), id_X) \end{aligned}$$

Thus, the proof is reduced to showing that $(X, f!(E, F, \varphi), id_X) \sim (M, E, f)$. We would like to use Lemma 3.8. However, in general, f need not be an embedding. To circumvent this problem, let $e : M \hookrightarrow S$ be an embedding of S into a sphere of even (resp. odd) dimension if M is even (resp. odd) dimensional. Then

$$(f, e) : M \hookrightarrow X \times S \text{ and } (id_X, 0) : X \hookrightarrow X \times S$$

are embeddings. Moreover, (f, e) is homotopic to $(f, 0)$. Using Lemma 3.8 (twice), we obtain

$$\begin{aligned} (M, E, f) &\sim (X \times S, (f, e)![E, F, \varphi], proj_X) \\ &\sim (X \times S, (f, 0)![E, F, \varphi], proj_X) \\ &\sim (X, f![E, F, \varphi], id_X) \end{aligned}$$

and hence the desired result. □

Theorem 3.10. *If (X, Y) is a finite CW-pair, then α is an isomorphism.*

Proof. The previous lemma implies that α has a right inverse (namely, β). To see that β is also a right inverse, consider the following commutative diagram:

$$\begin{array}{ccc} K_*(X, Y, \phi) & \xrightarrow{h_*} & K_*(Z, \partial Z; \phi) \\ \downarrow \beta \circ \alpha & & \downarrow id \\ K_*(X, Y; \phi) & \xrightarrow{h_*} & K_*(Z, \partial Z; \phi) \end{array}$$

where we have used the previous lemma to obtain that $\beta \circ \alpha : \bar{K}_*(Z, \partial Z; \phi) \rightarrow \bar{K}_*(Z, \partial Z; \phi)$ is equal to the identity map. Hence, the map $\beta \circ \alpha : \bar{K}_*(X, Y; \phi) \rightarrow \bar{K}_*(X, Y; \phi)$ is the restriction of the identity and hence is the identity map. \square

Proposition 3.11. *If (X, Y) is a finite CW-pair, then the following sequence is exact:*

$$\begin{array}{ccccc} \bar{K}_0(Y; \phi) & \xrightarrow{\bar{\iota}} & \bar{K}_0(X; \phi) & \xrightarrow{\bar{r}} & \bar{K}_0(X, Y; \phi) \\ \uparrow \bar{\delta} & & & & \downarrow \bar{\delta} \\ \bar{K}_1(X, Y; \phi) & \xleftarrow{\bar{r}} & \bar{K}_1(X; \phi) & \xleftarrow{\bar{\iota}} & \bar{K}_1(X; \phi) \end{array}$$

Proof. This follows from the fact that α is a natural isomorphism and the corresponding exact sequence in KK -theory. \square

To prove the result without α is rather difficult. However, it follows from the bordism relations in the various groups that the composition of successive maps are zero. For example,

$$(i_* \circ \partial)(W, [(E, F, \varphi)], f) = (\partial W, [(E, F, \varphi)]|_{\partial W}, i \circ f|_{\partial W})$$

is the boundary of the bordism (in $\bar{K}_*(X; \phi)$) given by $(W, [(E, F, \varphi)], f)$. From this point on, we only consider the absolute case (i.e., the case when $Y = \emptyset$). The reader can provide the details in the general case if they are so inclined.

Theorem 3.12. *If X is a finite CW-complex, then the following sequence is exact:*

$$\begin{array}{ccccc} K_0(X; B_1) & \xrightarrow{\phi_*} & K_0(X; B_2) & \xrightarrow{\bar{r}} & \bar{K}_1(X; \phi) \\ \uparrow \bar{\delta} & & & & \downarrow \bar{\delta} \\ \bar{K}_0(X; \phi) & \xleftarrow{\bar{r}} & K_1(X; B_2) & \xleftarrow{\phi_*} & K_1(X; B_1) \end{array}$$

where the maps are defined as follows:

- (1) $\phi_* : K_*(X; B_1) \rightarrow K_*(X; B_2)$ takes a B_1 -cycle (M, F, f) to the B_2 -cycle $(M, \phi_*(F), f)$.
- (2) $\bar{r} : K_*(X; B_2) \rightarrow \bar{K}_{*+1}(X; \phi)$ takes a cycle (M, E, f) to $(M \times S^1, \hat{r}(u), f \circ \pi)$ where

$$\begin{aligned} u : \pi^*(E) &\rightarrow \pi^*(E) \\ ((m, z), e) &\mapsto ((m, z), z \cdot e) \end{aligned}$$

and \hat{r} is the map in the exact sequence in the statement of Theorem 2.2.

- (3) $\bar{\delta} : \bar{K}_*(X; \phi) \rightarrow K_*(X; B_1)$ takes a cycle $(M, [(E, F, \alpha)], f)$ to $(M, [E] - [F], f)$.

Proof. Theorem 3.10 and [28, Theorem 2.3.3] imply that

$$K_*(X; B_i) \cong KK^*(C(X), B_i) \text{ and } K_*(X; \phi) \cong KK^{*-1}(C(X), C_\phi)$$

via the explicit maps

$$\mu_{B_i} : K_*(X; B_i) \rightarrow KK^*(C(X), B_i) \text{ and } \alpha : K_*(X; \phi) \rightarrow KK^*(C(X), SC_\phi)$$

This reduces the proof to showing that the following diagram commutes:

$$\begin{array}{ccccccc} \rightarrow & K_*(X; B_1) & \rightarrow & K_*(X; B_2) & \rightarrow & \bar{K}_*(X; \phi) & \rightarrow & K_{*+1}(X; B_1) & \rightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \rightarrow & KK^*(C(X), B_1) & \rightarrow & KK^*(C(X), B_2) & \rightarrow & KK^*(C(X), C_\phi) & \rightarrow & KK^{*+1}(C(X), B_1) & \rightarrow \end{array}$$

That $\mu_{B_1} \circ \phi_* = \phi_* \circ \mu_{B_2}$ follows from [28, Proposition 2.2.10]. The proof that $\mu_{B_1} \circ \delta = \delta \circ \alpha$ is as follows. Using the fact the PD_M is implemented via an explicit KK -class and basic properties of KK -theory, we have that

$$\begin{aligned} (\delta \circ \alpha)(M, [E, F, \varphi], f) &= \delta(f_*(PD_M([E, F, \varphi]))) \\ &= (ev_0)_*(f_*(PD_M([E, F, \varphi]))) \\ &= f_*(PD_M((ev_0)_*([E, F, \varphi]))) \\ &= \mu_{B_1}(M, (ev_0)_*[E, F, \varphi], f) \\ &= (\mu_{B_1} \circ \bar{\delta})(M, [E, F, \varphi], f) \end{aligned}$$

Finally, we must show that $\hat{r} \circ \mu_{B_2} = \alpha \circ \bar{r}$. Let (M, E_{B_2}, f) be a cycle in $K_*(X; B_2)$. Then, again using the fact the PD_M is implemented via an explicit KK -class and basic properties of KK -theory, we obtain

$$\begin{aligned} (\hat{r} \circ \mu_{B_2})(M, E_{B_2}, f) &= \hat{r}(f_*(PD_M([E_{B_2}]))) \\ &= \hat{r}((f \circ \pi_{S^1})_*(PD_{M \times S^1}(\pi_{S^1}^*(E_{B_2}) \otimes u_{S^1}))) \\ &= (f_* \circ \pi_{S^1})_*(PD_{M \times S^1}(\hat{r}(\pi_{S^1}^*(E_{B_2}) \otimes u_{S^1}))) \\ &= (\alpha \circ \bar{r})(M, E_{B_2}, f) \end{aligned}$$

where the reader should note that u_{S^1} is defined in the statement of the theorem (in particular, see the definition of r). \square

4. BORDISM TYPE MODEL FOR $KK(C(X), C_\phi)$

4.1. Cycles and Relations. Based on the discussion in the introduction, we need different geometric cycles to model certain constructions in geometric K -homology. As such, in this section, we introduce an abelian group, $K_*(X; \phi)$, with the main result that this group fits into a six-term exact sequence (see Theorem 4.19). In fact, we define two types of cycles. The first uses vector bundle data while the second uses K -theory data; a detailed development is only given in the case of the second model as the two cases are rather similar.

Definition 4.1. Cycles with vector bundle data

A cycle (over X with respect to ϕ) is given by, $(W, (E_{B_2}, F_{B_1}, \alpha), f)$, where

- (1) W is a smooth, compact $spin^c$ -manifold with boundary;
- (2) E_{B_2} is a finitely generated projective Hilbert B_2 -module bundle over W ;
- (3) F_{B_1} is a finitely generated projective Hilbert B_1 -module bundle over ∂W ;
- (4) $\alpha : E_{B_2}|_{\partial W} \cong \phi_*(F_{B_1})$ is an isomorphism of Hilbert B_2 -module bundles;
- (5) $f : W \rightarrow X$ is a continuous map.

Definition 4.2. Cycles with K -theory data

A cycle (over X with respect to ϕ) is a triple, (W, ξ, f) , where:

- (1) W is a smooth, compact $spin^c$ -manifold with boundary;
- (2) $\xi \in K^0(W, \partial W; \phi)$;
- (3) $f : W \rightarrow X$ is a continuous map.

The manifold, W , in a cycle need not be connected. As such, a cycle is called even (resp. odd) if each of its connected components are even (resp. odd) dimensional. We also let $\xi_{\partial W}$ and ξ_W denote the images of ξ under the maps $p_1 : K^0(W, \partial W; \phi) \rightarrow K^0(\partial W; B_1)$ and $p_2 : K^0(W, \partial W; \phi) \rightarrow K^0(W; B_2)$ respectively. The opposite of a cycle, (W, ξ, f) , is the same data only W is given the opposite $spin^c$ -structure. It is denoted by $-(W, \xi, f)$. The disjoint union of cycles, (W, ξ, f) and $(\tilde{W}, \tilde{\xi}, \tilde{f})$, is given by the cycle:

$$(W \dot{\cup} \tilde{W}, \xi \dot{\cup} \tilde{\xi}, f \dot{\cup} \tilde{f})$$

Two cycles, (W, ξ, f) and $(\tilde{W}, \tilde{\xi}, \tilde{f})$, are isomorphic if there exists a diffeomorphism, $h : W \rightarrow \tilde{W}$, such that h preserves the $spin^c$ -structure, $h^*(\tilde{\xi}) = \xi$, and $\tilde{f} \circ h = f$. Throughout, a “cycle” more precisely refers to an isomorphism class of a cycle.

Definition 4.3. A regular domain, Y , of a manifold M is a closed submanifold of M such that

- (1) $\text{int}(Y) \neq \emptyset$;
- (2) If $p \in \partial Y$, then there exists a coordinate chart, $\phi : U \rightarrow \mathbb{R}^n$ centered at p such that $\phi(Y \cap U) = \{x \in \phi(U) | x_n \geq 0\}$.

Definition 4.4. A bordism (with respect to X and ϕ) is given by (Z, W, η, F) where

- (1) Z is a compact $spin^c$ -manifold;
- (2) $W \subseteq \partial Z$ is a regular domain;
- (3) $\eta \in K^0(Z, \partial Z - \text{int}(W); \phi)$;
- (4) $g : Z \rightarrow X$ is a continuous map.

Remark 4.5. The “boundary” of a bordism, (Z, W, η, F) , is given by $(W, \eta|_W, g|_W)$. The fact that W is a regular domain of ∂Z ensures the boundary is indeed a cycle in $K_*(X; \phi)$. Moreover, if (W, ξ, f) is a boundary in the sense of Definition 4.4, then $(\partial W, \xi_{B_1}, f|_{\partial W})$ is a boundary as a cycle in $K_*(X; B_1)$.

Definition 4.6. Two cycles are bordant if there exists bordism with boundary given by $(W, \xi, f) \dot{\cup} -(W', \xi', f')$. This relation is denoted by

$$(W, \xi, f) \sim_{bor} (W', \xi', f')$$

Proposition 4.7. The operation \sim_{bor} is an equivalence relation.

Proof. We begin with reflexivity. Results in [12, Sections I.3 and I.4] imply that there exists a smooth $spin^c$ -manifold with boundary, Z , which is homeomorphic (via h) to $W \times [0, 1]$. Moreover, $W \dot{\cup} -W$ is a regular domain of the boundary of this manifold. Thus, if $\pi_W : W \times [0, 1] \rightarrow W$ is the projection onto W , then

$$(Z, W \dot{\cup} -W, h^*(\pi_W^*(\xi)), f \circ \pi_W \circ h)$$

forms a bordism. Moreover, it has boundary (in the sense of Definition 4.4) $(W, \xi, f) \dot{\cup} -(W, \xi, f)$. In other words,

$$(W, \xi, f) \sim_{bor} (W, \xi, f)$$

That bordism is symmetric is trivial.

For transitivity, let $\{(W_i, \xi_i, f_i)\}_{i=0}^2$ be cycles and $(Z_0, W_0 \dot{\cup} -W_1, \nu_0, F_0)$ and $(Z_1, W_1 \dot{\cup} -W_2, \nu_1, F_1)$ be bordisms from (W_0, ξ_0, f_0) to (W_1, ξ_1, f_1) and (W_1, ξ_1, f_1) and (W_2, ξ_2, f_2) respectively. Then, by “straightening the angle” (see [12]), $Z_0 \cup_{W_1} Z_1$ can be given the structure of a smooth $spin^c$ -manifold. Also, let $F = F_0 \cup_{W_1} F_1$ and $\nu = \nu_0 \cup_{W_1} \nu_1$. Then, $(Z, W_0 \dot{\cup} W_2, \nu, F)$ forms a bordism from (W_0, ξ_0, f_0) to (W_2, ξ_2, f_2) . \square

Definition 4.8. Let (W, ξ, f) be a cycle and E a $spin^c$ -vector bundle of even rank over W . Then the vector bundle modification of (W, ξ, f) by E is defined to be:

$$(W^E, \pi^*(\xi) \otimes_{\mathbb{C}} \beta_E, f \circ \pi)$$

where

- (1) $\mathbf{1}$ is the trivial real line bundle over W (i.e., $W \times \mathbb{R}$);
- (2) $W^E = S(E \oplus \mathbf{1})$ (i.e., the sphere bundle of $E \oplus \mathbf{1}$);
- (3) β_E is the “Bott element” in $K^0(W^E)$ (see [25, Section 2.5] for the construction of this element);
- (4) $\otimes_{\mathbb{C}}$ denotes the $K^0(W^E)$ -module structure of $K^0(W^E, \partial W^E; \phi)$;
- (5) $\pi : W^E \rightarrow W$ is the bundle projection.

The vector bundle modification of (W, ξ, f) by E is often denoted by $(W, \xi, f)^E$.

Remark 4.9. If (W, ξ, f) is a cycle and E is a $spin^c$ -vector bundle of even rank over W , then $(\partial W, \xi_{B_1}, f|_{\partial W})^{E|_{\partial W}} = \partial(W, \xi, f)^E$.

Definition 4.10. Let \sim be the equivalence relation generated by bordisms and vector bundle modification (i.e., $(W, \xi, f) \sim (W, \xi, f)^E$, for any even rank $spin^c$ -vector bundle, E , over W). Also let

$$K_*(X; \phi) = \{(W, \xi, f)\} / \sim$$

The grading is given as follows. A cycle (W, ξ, f) is said to be even (resp. odd) if the connected components of W are all even (resp. odd) dimensional. Then, $K_0(X; \phi)$ is even cycles modulo \sim and $K_1(X; \phi)$ is likewise only with odd cycles. Note that the relation \sim preserves this grading.

Proposition 4.11. Let (W, ξ_1, f) and (W, ξ_2, f) be ϕ -cycles. Then

$$(W, \xi_1, f) \dot{\cup} (W, \xi_2, f) \sim (W, \xi_1 \oplus \xi_2, f)$$

Proof. The proof of this proposition is similar to the proof of Proposition 4.3.2 in [25]. A vector bundle modification (by a trivial bundle) implies that

$$\begin{aligned} (W, \xi_1, f) \dot{\cup} (W, \xi_2, f) &\sim (W \times S^2, \pi^*(\xi_1) \otimes \beta, f \circ \pi) \dot{\cup} (W \times S^2, \pi^*(\xi_2) \otimes \beta, f \circ \pi) \\ (W, \xi_1 \oplus \xi_2, f) &\sim (W \times S^2, \pi^*(\xi_1 \oplus \xi_2) \otimes \beta, f \circ \pi) \end{aligned}$$

where β is the Bott element and $\pi : W \times S^2 \rightarrow W$ is the projection map.

In the proof of Proposition 4.3.2 in [25], an explicit bordism between (S^2, β) and $(S^2, \beta) \dot{\cup} (S^2, \beta)$ is constructed; crossing this bordism with W and “straightening the angle” leads to

$$(W \times S^2, \pi^*(\xi_1) \otimes \beta, f \circ \pi) \dot{\cup} (W \times S^2, \pi^*(\xi_2) \otimes \beta, f \circ \pi) \sim_{bor} (W \times S^2, \pi^*(\xi_1 \oplus \xi_2) \otimes \beta, f \circ \pi)$$

and hence the desired result. \square

Proposition 4.12. $K_*(X; \phi)$ with the operation of disjoint union is an abelian group. The unit is given by the trivial (i.e., empty) cycle and the inverse of a cycle is given its opposite.

Proof. It is clear that disjoint union gives $K_*(X; \phi)$ the structure of an abelian semigroup. The proof that bordisms is an equivalence relation (i.e., Theorem 4.7) implies both that $K_*(X; \phi)$ is a group and the unit and inverses are given as in the statement of the theorem. \square

4.2. Normal bordisms and six-term exact sequence.

Definition 4.13. Let E be a vector bundle. A vector bundle, F , is called a complementary bundle for E , if $E \oplus F$ is a trivial vector bundle. If M is a manifold (possibly with boundary), then a complementary bundle for TM will be called a normal bundle.

Given a manifold with boundary $(W, \partial W)$, we can produce a normal bundle by taking a neat embedding in $H^n := \{(v_1, \dots, v_n) \in \mathbb{R}^n \mid v_1 \geq 0\}$.

Definition 4.14. Let (W, ξ, f) and (W', ξ', f') be two cycles in $K_*(X; \phi)$. Then, there is a normal bordism from (W, ξ, f) to (W', ξ', f') if there exists normal bundles $N_W, N_{W'}$ (over W and W' respectively) such that

$$(W, \xi, f)^{N_W} \sim_{bor} (W', \xi', f')^{N_{W'}}$$

This relation is denoted by \sim_{nor} .

The next lemma is standard (see for example [25, Lemma 4.5.7]), while the one following it is a natural generalization of [25, Lemma 4.4.3]. The proof of the latter is left to the reader.

Lemma 4.15. Normal bundles are stably isomorphic (i.e., if N_1 and N_2 are normal bundles for W , then there exists trivial bundles, E_1 and E_2 , such that $N_1 \oplus E_1 \cong N_2 \oplus E_2$).

Lemma 4.16. If E_1 and E_2 are $spin^c$ -vector bundles with even dimensional fibers over a compact $spin^c$ -manifold W , then, for any $\xi \in K^0(W, \partial W; \phi)$ and $f : W \rightarrow X$,

$$(W, \xi, f)^{E_1 \oplus E_2} \sim_{bor} ((W, \xi, f)^{E_1})^{p^*(E_2)}$$

where $p : S(E_1 \oplus \mathbf{1}) \rightarrow W$ is the projection map.

Proposition 4.17. Let X be a finite CW-complex. Then, the relation of normal bordism of cycles is an equivalence relation. Moreover, it is equal to the relation constructed in Definition 4.10.

Proof. We leave it to the reader to verify that \sim_{nor} is reflexive and symmetric.

To show \sim_{nor} is transitive, let $\{(W_i, \xi_i, f_i)\}_{i=0}^2$ be cycles. Moreover, assume that for each $i = 0, 1, 2$, N_i is a normal bundle for W_i , and N'_1 is a normal bundle for W_1 such that

$$\begin{aligned} (W_0, \xi_0, f_0)^{N_0} &\sim_{bor} (W_1, \xi_1, f_1)^{N_1} \\ (W_1, \xi_1, f_1)^{N'_1} &\sim_{bor} (W_2, \xi_2, f_2)^{N_2} \end{aligned}$$

Lemma 4.15 implies that there are trivial bundles ϵ_1 and ϵ'_1 such that

$$N_1 \oplus \epsilon_1 \cong N'_1 \oplus \epsilon'_1$$

Let ϵ_0 and ϵ_2 be trivial bundles (over W_0 and W_2 respectively) of the same rank as ϵ_1 and ϵ'_1 respectively. Following the notation in Lemma 4.16 for the bundle projections and, using the fact that trivial bundles extend across bordisms, we find that

$$\begin{aligned} ((W_0, \xi_0, f_0)^{N_0})^{p_0^*(\epsilon_0)} &\sim_{bor} ((W_1, \xi_1, f_1)^{N_1})^{p_1^*(\epsilon_1)} \\ ((W_1, \xi_1, f_1)^{N'_1})^{p_{1'}^*(\epsilon'_1)} &\sim_{bor} ((W_2, \xi_2, f_2)^{N_2})^{p_2^*(\epsilon_2)} \end{aligned}$$

Moreover, Lemma 4.16 and $N_1 \oplus \epsilon_1 \cong N'_1 \oplus \epsilon'_1$ imply that

$$\begin{aligned} (W_0, \xi_0, f_0)^{N_0 \oplus \epsilon_0} &\sim_{bor} (W_1, \xi_1, f_1)^{N_1 \oplus \epsilon_1} \\ &\sim_{bor} (W_1, \xi_1, f_1)^{N'_1 \oplus \epsilon'_1} \\ &\sim_{bor} (W_2, \xi_2, f_2)^{N_2 \oplus \epsilon_2} \end{aligned}$$

Transitivity of \sim_{nor} then follows since $N_0 \oplus \epsilon_0$ and $N_2 \oplus \epsilon_2$ are normal bundles for W_0 and W_2 respectively.

The proof will be complete upon showing \sim and \sim_{nor} are the same relation. That \sim_{nor} is a weaker relation than \sim is clear. On the other hand, we must show that

- (1) If (W, ξ, f) is a cycle and E is a smooth $spin^c$ -vector bundle with even-dimensional fibers over W , then $(W, \xi, f)^E \sim_{nor} (W, \xi, f)$.
- (2) Any boundary in the sense of Definition 4.4 also normally bounds.

For Item 1, let $p_E : W^E \rightarrow W$ denote the projection map, N a normal bundle for W , and E^c a complement to E (i.e., $E \oplus E^c$ is a trivial bundle). Then (as the reader can verify) $p_E^*(E^c \oplus N \oplus \mathbf{1})$ is a normal bundle for W^E .

Moreover, using Lemma 4.16, we obtain

$$((W, \xi, f)^E)^{p_E^*(E^c \oplus N \oplus \mathbf{1})} \sim_{bor} (W, \xi, f)^{E \oplus E^c \oplus N \oplus \mathbf{1}}$$

The result (i.e., Item 1 above) follows upon noticing that $E \oplus E^c \oplus N \oplus \mathbf{1}$ is a normal bundle for W .

For Item 2, let (W, ξ, f) be the boundary of a bordism (Z, W, ν, F) . Let N be a normal bundle for Z . Then

$$\partial(Z, W, \nu, F)^{N \oplus \mathbf{1}} = (W, \xi, f)^{N|_W \oplus \mathbf{1}}$$

$N|_W \oplus \mathbf{1}$ is a normal bundle for W ; hence (W, ξ, f) normally bounds. \square

Corollary 4.18. *Let X be a finite CW-complex. Then, a cycle in $K_*(X; \phi)$ is trivial if and only if it normally bounds.*

Theorem 4.19. *If X is a finite CW-complex, then the following sequence is exact:*

$$\begin{array}{ccccc} K_0(X; B_1) & \xrightarrow{\phi_*} & K_0(X; B_2) & \xrightarrow{r} & K_0(X; \phi) \\ \uparrow \delta & & & & \downarrow \delta \\ K_1(X; \phi) & \xleftarrow{r} & K_1(X; B_2) & \xleftarrow{\phi_*} & K_1(X; B_1) \end{array}$$

where the maps are defined as follows:

- (1) $\phi_* : K_*(X; B_1) \rightarrow K_*(X; B_2)$ takes a B_1 -cycle (M, F, f) to the B_2 -cycle $(M, \phi_*(F), f)$.
- (2) $r : K_*(X; B_2) \rightarrow K_*(X; \phi)$ takes a cycle (M, E, f) to $(M, (E, \emptyset, \emptyset), f)$.
- (3) $\delta : K_*(X; \phi) \rightarrow K_{*+1}(X; B_1)$ takes a cycle $(W, (E, F, \alpha), f)$ to $(\partial W, F, f|_{\partial W})$.

Proof. That the maps are well-defined is clear in the case of ϕ_* and follows from Remarks 4.5 and 4.9 in the case of r and δ .

The bordism relations on the various cycles imply that the composition of successive maps is zero. The details in the case of $r \circ \phi_*$ are as follows. Let (M, ξ, f) be a B_1 -cycle and $\pi_M : M \times [0, 1] \rightarrow M$ be the projection onto M . Then $(M \times [0, 1], \pi_M^*(\xi_{B_1}), f \circ \pi_M)$ is a bordism between (M, ξ_{B_1}, f) and its opposite. Moreover, this produces a bordism in $K_*(X; \phi)$ between $(M, \phi_*(\xi_{B_1}), f)$ and the empty cycle as follows.

In the notation of Definition 4.5, let

$$Z = M \times [0, 1], \eta = \phi_*(\pi_M^*(\xi_{B_1})), g = f \circ \pi_M$$

For the regular domain in this bordism, we take M . As such, $\partial Z - \text{int}(M)$ is $-M$ and the “boundary” of this bordism is

$$(M, \phi_*(\xi_{B_1}), f)$$

Hence $(M, \phi_*(\xi_{B_1}), f)$ is trivial in $K_*(X; \phi)$. We leave it to the reader to show that $\phi_* \circ \delta$ and $\delta \circ r$ are both zero.

We are left to show that

$$\ker(\phi_*) \subseteq \text{im}(\delta), \ker(\delta) \subseteq \text{im}(r), \ker(r) \subseteq \text{im}(\phi_*)$$

With the goal of proving $\ker(\phi_*) \subseteq \text{im}(\delta)$, fix (M, ξ_{B_1}, f) such that $(M, \phi_*(\xi_{B_1}), f)$ is trivial in $K_*(X; B_2)$. By [25, Corollary 4.5.16], there exists a normal bundle, N , such that

$$(M, \phi_*(\xi_{B_1}), f)^N \cong \partial(W, \eta_{B_2}, g)$$

where (W, η_{B_2}, g) is a B_2 -cycle with boundary. By construction, (η_{B_2}, ξ_{B_1}) is an element of the pullback along the maps $K^0(X; B_2) \rightarrow K^0(Y; B_2)$ and $K^0(Y; B_1) \rightarrow K^0(Y; B_2)$. Proposition 2.6 implies that there exists $\xi \in K^0(W, \partial W; \phi)$ which maps (under the natural maps) to $(\eta_{B_2}, \xi_{B_1}) \in K^0(W; B_2) \oplus K^0(\partial W; B_1)$. In particular, (W, ξ, g) forms a cycle in $K_{*-1}(X; \phi)$. Moreover,

$$\delta(W, \xi, g) = (M, \phi_*(\xi_{B_1}), f)^N \sim (M, \phi_*(\xi_{B_1}), f)$$

To show that $\ker(\delta) \subseteq \text{im}(r)$, let (W, ξ, f) be a cycle such that $(\partial W, \xi_{B_1}, f|_{\partial W})$ is trivial in $K_*(X; B_1)$. By [25, Corollary 4.5.16], there exists normal bundle, N over ∂W , such that

$$(\partial W, \xi_{B_1}, f|_{\partial W})^N = \partial(Z, \eta, g)$$

The normal bundle N may not extend to W . However, by Lemma 4.15, if N' is a normal bundle for W , there exists a trivial bundle ϵ over ∂W such that $N \oplus \epsilon \cong N'|_{\partial W}$. Hence,

$$\begin{aligned} \delta((W, \xi, f)^{N'}) &= (\partial W, \xi_{B_1}, f)^{N'|_{\partial W}} \\ &= (\partial W, \xi_{B_1}, f)^{N \oplus \epsilon} \\ &\sim_{bor} ((\partial W, \xi_{B_1}, f)^N)^{p^*(\epsilon)} \\ &= \partial((Z, \eta, g)^{\epsilon_Z}) \end{aligned}$$

where $p : \partial W^N \rightarrow \partial W$ is the projection map and ϵ_Z is the trivial vector bundle over Z with fiber dimension the same as ϵ . Summarizing, from a cycle, (W, ξ, f) , which is in the kernel of δ , we have produced an equivalent cycle (i.e., $(W, \xi, f)^{N'}$) whose image under δ is a boundary (rather than just trivial).

Hence, without loss of generality, assume that $(\partial W, \xi_{B_1}, f|_{\partial W})$ is a boundary in $K_{*-1}(X; B_1)$. Let (Z, η, g) be a B_1 -cycle with boundary such that

$$(\partial W, \xi_{B_1}, f|_{\partial W}) = \partial(Z, \eta, g)$$

Then $(Z, \phi_*(\eta), g)$ is a B_2 -cycle with boundary. Form the closed (smooth, $spin^c$) manifold $\tilde{W} = W \cup_{\partial W} Z$. As the reader will note, the K -theory data and continuous function are compatible along ∂W . Hence, we can form the B_2 -cycle, $(\tilde{W}, \xi_{B_2} \cup \phi_*(\eta), f \cup g)$.

It remains to show that

$$r(\tilde{W}, \xi_{B_2} \cup \phi_*(\eta), f \cup g) \sim (W, \xi, f)$$

This result follows from the following bordism in the $K_*(X; \phi)$:

$$(\tilde{W} \times [0, 1], \tilde{W} \dot{\cup} W, \tilde{\xi}, (f \cup g) \circ \pi)$$

where $\pi : \tilde{W} \times [0, 1] \rightarrow \tilde{W}$ and $\tilde{\xi}$ is the element in $K^0(\tilde{W} \times [0, 1], Z; \phi)$ formed from $\pi^*(\xi_{B_2} \cup \phi^*(\eta)) \in K^0(\tilde{W} \times [0, 1]; B_2)$ and $\eta_{B_1} \in K^0(Z; B_1)$.

Finally, we show that $\ker(r) \subseteq \text{im}(\phi_*)$. Let (M, ξ_{B_2}, f) be a cycle in $K_*(X; B_2)$ which is mapped to the trivial element in $K_*(X; \phi)$. By Lemma 4.18, there exists a normal bundle, N , such that

$$(M, \xi_{B_2}, f)^N = \partial(Z, M^N, \eta, g)$$

Consider (Z, η_{B_2}, g) as a bordism in $K_*(X; B_2)$. It has boundary

$$(M, \xi_{B_2}, f)^N \dot{\cup} (\partial Z - M, \phi_*(\eta_{B_1}), g|_{\partial Z - M})$$

The bordism relation in $K_*(X; B_2)$ then implies that

$$(M, \xi_{B_2}, f)^N \sim \phi_*((\partial Z - M, (\eta_{B_1}), g|_{\partial Z - M}))$$

This completes the proof. \square

5. ISOMORPHISM FROM $\bar{K}_*(X; \phi)$ TO $K_*(X; \phi)$

Definition 5.1. Let $\mu_\phi : \bar{K}_*(X; \phi) \rightarrow K_{*+1}(X; \phi)$ be the map defined on cycles via

$$(M, [E, F, \varphi], f) \mapsto (M \times [0, 1], [(\mathcal{E}_{B_2}, \mathcal{F}_{B_1}, \alpha)], f \circ \pi)$$

where

- (1) $\pi : M \times [0, 1] \rightarrow M$ is the projection map;
- (2) \mathcal{E}_{B_2} is the Hilbert B_2 -module bundle, $\phi_*(\pi^*(E))$;
- (3) \mathcal{F}_{B_1} is the Hilbert B_1 -module bundle defined by taking its fiber at $M \times \{0\}$ to be E and its fiber at $M \times \{1\}$ to be F ;
- (4) $\alpha : \phi_*(\mathcal{F}_{B_1}) \rightarrow \mathcal{E}_{B_2}|_{M \times \{0\} \dot{\cup} M \times \{1\}}$ given by the identity on $M \times \{0\}$ and φ on $M \times \{1\}$;

Since the definition of μ_ϕ involves the choice of cocycle (rather than just the class in $K^0(X; \phi)$), our first goal is to show that μ_ϕ is well-defined at the level of cycles. As such, we must show that

- (1) If $(E, F, \varphi) \cong (E', F', \varphi')$, then $\mu_\phi(M, [E, F, \varphi], f) = \mu_\phi(M, [E', F', \varphi'], f)$.
- (2) If (E, F, φ) is an elementary cocycle, then $\mu_\phi(M, [E, F, \varphi], f)$ is trivial.

The first item follows from the bordism relation in $K_*(X; \phi)$. Let (β_1, β_2) be an isomorphism from (E, F, φ) to (E', F', φ) ; the definition of isomorphism in this context is given in Section 2.1. Also, let D denote the closed unit disk and

$$I_1 = \left[0, \frac{\pi}{2}\right], \quad I_2 = \left[\frac{\pi}{2}, \pi\right], \quad I_3 = \left[\pi, \frac{3\pi}{2}\right], \quad I_4 = \left[\frac{3\pi}{2}, 0\right]$$

be intervals inside $\partial D = S^1$. Consider the following bordism in $K_*(X; \phi)$

$$\left(M \times D, M \times \left[\frac{\pi}{2}, \frac{3\pi}{4}\right] \dot{\cup} M \times \left[\frac{5\pi}{4}, \frac{3\pi}{2}\right], (\hat{\mathcal{E}}_{B_2}, \hat{\mathcal{F}}_{B_1}, \hat{\alpha}), f \circ \pi\right)$$

where

- (1) $\hat{\mathcal{E}}_{B_2}$ is formed on $M \times S^1$ by respectively clutching bundles $\pi_{I_1}^*(F) \otimes_{\phi} B_2$, $\pi_{I_2}^*(E) \otimes_{\phi} B_2$, $\pi_{I_3}^*(E') \otimes_{\phi} B_2$, and $\pi_{I_4}^*(F') \otimes_{\phi} B_2$ along $M \times \{\frac{\pi}{2}\}$, $M \times \{\pi\}$, $M \times \{\frac{3\pi}{2}\}$, and $M \times \{0\}$ via the isomorphisms φ , $\phi^*(\beta_1)$, φ' , and $\phi^*(\beta_2)$. The reader should note that this bundle extends to all of $M \times D$ since it is formed by taking the homotopy associated to the isomorphism $\phi_*(\beta_1)$ and then straightening the angle;
- (2) $\hat{\mathcal{F}}_{B_1}$ is given by $\pi_{I_1}^*(F) \cup_{\beta_2} \pi_{I_4}^*(F')$ on $M \times [\frac{3\pi}{2}, \frac{\pi}{2}]$ and $\pi_{I_1}^*(E) \cup_{\beta_1} \pi_{I_2}^*(E')$ on $M \times [\frac{3\pi}{4}, \frac{5\pi}{4}]$;
- (3) $\hat{\alpha}$ is defined by φ on fibers at $M \times \{\frac{\pi}{2}\}$, φ' on fibers at $M \times \{\frac{3\pi}{2}\}$ and the identity at fibers $M \times \{\frac{3\pi}{4}\}$ and $M \times \{\frac{5\pi}{4}\}$.

As the reader can verify, this bordism has the required boundary. Such a reader may wish to begin by checking the result in the case of $M = pt$.

The second item will also follow from the bordism relation. We form the required bordism by taking the regular domain $M \times [0, 1]$ inside $\partial(M \times D) = M \times S^1$. Then

$$((M \times D, M \times [0, 1]), (\phi_*(\pi^*(E)), \pi_{[0, 1]}^*(E), \alpha), f \circ \pi)$$

forms the required bordism where the construction of the bundle data is similar to the previous argument.

Next, the relations must be considered. For vector bundle modification, let $(M, [E, F, \varphi, f])$ be a \bar{K} -cycle and V be an even rank $spin^c$ -vector bundle over M . Then

$$\begin{aligned} \mu_{\phi}((M, [E, F, \varphi, f])^V) &= \mu_{\phi}(M^V, [\pi^*(E), \pi^*(F), \pi^*(\varphi)] \otimes \beta, f \circ \pi) \\ &= (M^V \times [0, 1], [(\pi^*(\mathcal{E}))_{B_2}, \pi^*(\mathcal{F})_{B_1}, \alpha_{\pi^*(\varphi)}], f \circ \pi) \\ &= (M \times [0, 1], [(\mathcal{E}_{B_2}, \mathcal{F}_{B_1}, \alpha)], f \circ \pi)^{\pi^*(V)} \\ &= \mu_{\phi}(M, [E, F, \varphi, f])^{\pi^*(V)} \\ &\sim \mu_{\phi}(M, [E, F, \varphi, f]) \end{aligned}$$

For the bordism relation, let $(M, [E, F, \varphi, f])$ be the boundary of $(\tilde{M}, [\tilde{E}, \tilde{F}, \tilde{\varphi}], \tilde{f})$. Using the straightening the angle technique, we have a smooth $spin^c$ -manifold, \tilde{Z} , which is homeomorphic to $\tilde{M} \times [0, 1]$. Let h denote such a homeomorphism. By construction, $\partial \tilde{Z}$ contains $M \times [0, 1]$ as a regular domain and

$$(4) \quad \partial \tilde{Z} - (M \times (0, 1)) = \tilde{M} \dot{\cup} - \tilde{M}$$

We form a bordism in $K_*(X; \phi)$ by taking

$$((\tilde{Z}, M \times [0, 1]), (\mathcal{E}_{B_2}, \mathcal{F}_{B_1}, \alpha), g)$$

where

- (1) $\pi : \tilde{M} \times [0, 1] \rightarrow \tilde{M}$ is the projection map;
- (2) $\mathcal{E}_{B_2} = h^*(\pi^*(\tilde{E}))$;
- (3) \mathcal{F}_{B_1} is given by \tilde{E} on the first copy of \tilde{M} and \tilde{F} on the second (see Equation 4);
- (4) α is given by the identity on the fibers of the first copy of \tilde{M} and φ on the second;
- (5) $g = \tilde{f} \circ \pi \circ h$.

Thus, μ_ϕ is well-defined as a map from $\bar{K}_*(X; \phi) \rightarrow K_*(X; \phi)$.

Theorem 5.2. *If X is a finite CW-complex, then μ_ϕ is an isomorphism.*

Proof. The Five Lemma and Theorems 3.12 and 4.19 reduce the proof to showing the commutativity of the diagram:

$$\begin{array}{ccccccccc}
\rightarrow & K_1(X; B_2) & \xrightarrow{\bar{r}} & \bar{K}_0(X; \phi) & \xrightarrow{\bar{\delta}} & K_0(X; B_1) & \xrightarrow{\phi_*} & K_0(X; B_2) & \xrightarrow{\bar{r}} & \bar{K}_1(X; \phi) & \rightarrow \\
& \parallel & & \mu_\phi \downarrow & & \parallel & & \parallel & & \mu_\phi \downarrow & \\
\rightarrow & K_1(X; B_2) & \xrightarrow{r} & K_1(X; \phi) & \xrightarrow{\delta} & K_0(X; B_1) & \xrightarrow{\phi_*} & K_0(X; B_2) & \xrightarrow{r} & K_0(X; \phi) & \rightarrow
\end{array}$$

where the maps are defined in Definition 5.1 and Theorems 3.12 and 4.19.

The only nontrivial part is proving that $\mu_\phi \circ \bar{r} = r$. Let (M, E_{B_2}, f) be a cycle in $K_*(X; B_2)$. There are two steps in our proof. Firstly, we show that $(\mu_\phi \circ \bar{r})(M, E_{B_2}, f)$ is in the image of r and then show it is equal to $r(M, E_{B_2}, f)$. For the first step, by the definition of \bar{r} and [21, Section 3.21], we have

$$\begin{aligned}
\bar{r}(M, E_{B_2}, f) &= (M \times S^1, \hat{r}(u), f \circ \pi) \\
&\sim (M \times S^1, (\pi^*(F_{B_1}), \pi^*(F_{B_1}), \hat{u}), f \circ \pi)
\end{aligned}$$

where $\hat{u} \sim u$ in $K^1(M \times S^1; B_2)$, F_{B_1} is a B_1 -module bundle, and $\pi : M \times S^1 \rightarrow M$ is the projection map. Hence

$$(\mu_\phi \circ \bar{r})(M, E_{B_2}, f) = (M \times S^1 \times [0, 1], (\tilde{\pi}^*(F_{B_1} \otimes_\phi B_2, \pi^*(F_{B_1}), \alpha_{\hat{u}}), f \circ \tilde{\pi}))$$

where $\alpha_{\hat{u}}$ is defined as in Definition 5. Moreover, the bordism relation implies that

$$\begin{aligned}
(M \times S^1 \times [0, 1], (\tilde{\pi}^*(F_{B_1} \otimes_\phi B_2, \pi^*(F_{B_1}), \alpha_{\hat{u}}), f \circ \tilde{\pi})) &\sim (M \times S^1 \times S^1, (V_{\hat{u}}, \emptyset, \emptyset), f \circ \hat{\pi}) \\
&= r(M \times S^1 \times S^1, V_{\hat{u}}, f \circ \hat{\pi})
\end{aligned}$$

where $V_{\hat{u}}$ is the B_2 -module bundle obtained by clutching the vector bundle $\pi^*(F_{B_1}) \otimes_\phi B_2$ along $M \times S^1$ via \hat{u} and $\tilde{\pi} : M \times S^1 \times [0, 1] \rightarrow M$ and $\hat{\pi} : M \times S^1 \times S^1 \rightarrow M$ are the natural projection maps. This completes the first step of the proof.

The second step is to show that $(M \times S^1 \times S^1, V_{\hat{u}}, f \circ \hat{\pi}) \sim (M, E_{B_2}, f)$. Using the fact that u and \hat{u} are equivalent in $K^1(M \times S^1; B_2)$ and the bordism and vector bundle modification relations, we obtain

$$\begin{aligned}
(M \times S^1 \times S^1, V_{\hat{u}}, f \circ \pi) &\sim (M \times S^1 \times S^1, V_u, f \circ \pi) \\
&\sim_{bor} (M \times S^2, (\pi')^*(E_{B_2}) \otimes \beta, f \circ \pi') \\
&\sim_{VBM} (M, E_{B_2}, f)
\end{aligned}$$

where $\pi' : M \times S^2 \rightarrow M$ is the projection map and (as above) β denotes the Bott bundle. \square

Example 5.3. K -homology with coefficients

Suppose that

$$0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 0$$

is a short exact sequence of abelian groups. Then we have an associated six-term exact sequence of K -homology with coefficients:

$$\begin{array}{ccccc} K_0(X; G_1) & \xrightarrow{\phi_*} & K_0(X; G_2) & \xrightarrow{r} & K_0(X; G_3) \\ \uparrow \delta & & & & \downarrow \delta \\ K_1(X; G_3) & \xleftarrow{r} & K_1(X; G_2) & \xleftarrow{\phi_*} & K_1(X; G_1) \end{array}$$

Such sequences are referred to as the Bockstein sequence associated to the exact sequence of abelian groups. Using the mapping cone of a $*$ -homomorphism, ϕ , and the six-term exact sequence in KK -theory often lead to such sequences. Prototypical examples of ϕ are given by the following inclusions

$$\mathbb{C} \rightarrow M_n(\mathbb{C}), \quad \mathbb{C} \rightarrow D, \quad \mathbb{C} \rightarrow N$$

where D is a UHF-algebra with K_0 -group, \mathbb{Q} , and N is a II_1 -factor (recall that $K_0(N) \cong \mathbb{R}$ and $K_1(N) \cong \{0\}$). In these cases, Theorems 3.12 and 4.19 produces the Bockstein sequence associated (respectively) to the following exact sequences of abelian groups:

$$\begin{aligned} 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/k\mathbb{Z} &\rightarrow 0 \\ 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} &\rightarrow 0 \\ 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} &\rightarrow 0 \end{aligned}$$

In other words, Theorems 3.12 and 4.19 produce geometric models (via the cycles in Definitions 3.1 and 4.2 respectively) for $K_*(X; \mathbb{Z}/k\mathbb{Z})$, $K_*(X; \mathbb{Q}/\mathbb{Z})$, and $K_*(X; \mathbb{R}/\mathbb{Z})$.

In the next section, we discuss geometric K -homology with coefficient in \mathbb{R}/\mathbb{Z} in detail. Before doing so, we discuss a number of generalizations of the constructions considered to this point.

The Baum-Douglas model has been generalized to the equivariant and families index settings (see [10, 11, 16, 28]). In a number of cases, the models constructed in this paper also have such generalizations. In the equivariant setting, in the case of compact Lie groups, one should replace our cycles with the natural analogue based on [28]. In the case of a discrete group which acts properly, one should replace our cycles with the natural analogue of the cycles in [10]. A generalization to actions of groupoids also seems possible, but much more involved. The interested reader should compare the cycles defined in [10] with those in [16] as a starting point.

Let D denote a unital C^* -algebra. Then a model for $KK(C(X), D \otimes C_\phi)$ can be obtained directly from our results. One simply notes that the main results in this paper can be applied to the $*$ -homomorphism $id_D \otimes \phi : D \otimes B_1 \rightarrow D \otimes B_2$. If D is a commutative C^* -algebra (i.e., $D = C(Y)$), then there is an alternative approach; the cycles in this theory are defined as follows.

Definition 5.4. A cycle (over X with respect to ϕ and Y) is given by a triple, $(W, (E_{B_2}, F_{B_1}, \alpha), f)$, where

- (1) W is a smooth, compact $spin^c$ -manifold with boundary;
- (2) E_{B_2} is a finitely generated projective Hilbert B_2 -module bundle over $W \times Y$;

- (3) F_{B_1} is a finitely generated projective Hilbert B_1 -module bundle over $\partial W \times Y$;
- (4) $\alpha : E_{B_2}|_{Y \times \partial W} \cong \phi_*(F_{B_1})$ is an isomorphism of Hilbert B_2 -module bundles;
- (5) $f : W \rightarrow X$ is a continuous map.

The relation on such cycles is the natural generalization of the relation discussed in Section 4. One can replace the bundle data in such a cycle with a class in $K^0(Y \times W, Y \times \partial W; \phi)$. We will make use of this model in the next section in the particular case of K -homology with coefficients in \mathbb{R}/\mathbb{Z} .

6. \mathbb{R}/\mathbb{Z} -VALUED INDEX THEORY

In this section, we specialize to the case of $\phi : \mathbb{C} \rightarrow N$ where N is a II_1 -factor and $\phi : \lambda \rightarrow \lambda \cdot I$. As discussed in Example 5.3, for this choice of ϕ , $K_*(X; \phi)$ is a realization of $K_*(X; \mathbb{R}/\mathbb{Z})$. The reader should compare our construction in Section 6.1 to [3, Section 5] and the pairings in Section 6.2 to the development in [20, Section 6].

6.1. The index map and K -theory with \mathbb{R}/\mathbb{Z} -coefficients. Let Y denote a compact Hausdorff space; a useful special case to consider is $Y = pt$. Following [3, Section 5], we begin with the observation that $K^*(Y; \mathbb{R}/\mathbb{Z}) \cong \text{coker}(q)$ where $q : K^*(Y; \mathbb{Q}) \rightarrow K^*(Y; \mathbb{Q}/\mathbb{Z}) \oplus K^*(Y; \mathbb{R})$ is the natural map. The group $\text{coker}(q)$ is of course graded; we denote the grading via $\text{coker}(q^*)$.

The goal of this subsection is the construction of an explicit isomorphism at the level of cycles from $K^*(Y; \mathbb{R}/\mathbb{Z})$ (modeled using cycles in Definition 5.4) to $\text{coker}(q)$. When $Y = pt$, this amounts to the construction of a map from cycles in $K_0(pt; \phi)$ to \mathbb{R}/\mathbb{Z} . Based on the definition of q , we must construct maps from such cycles to $K^0(Y; \mathbb{Q}/\mathbb{Z})$ and $K^0(Y; \mathbb{R})$ respectively. We will refer to the map in the case of $Y = pt$ as an index map.

Let (W, ξ) be a cycle in $K^0(Y; \mathbb{R}/\mathbb{Z})$ and $(\partial W, \xi_{\mathbb{C}})$ denote $\delta(W, \xi) \in K^0(Y)$ (recall that δ was defined in the statement of Theorem 4.19).

The construction of the map from (W, ξ) to $K^*(Y; \mathbb{Q}/\mathbb{Z})$ is as follows. Since $\phi_*(\partial W, \xi_{\mathbb{C}}) = 0$ and ϕ_* is rationally injective, there exists $k \in \mathbb{N}$, normal bundle N_W over W , and Baum-Douglas bordism (within the group $K^*(Y)$), $(Q, \eta_{\mathbb{C}})$, such that

$$k(\partial W, \xi_{\mathbb{C}})^N = \partial(Q, \eta_{\mathbb{C}})$$

where N denotes N_W restricted to ∂W . By construction, Q has the structure of a spin^c $\mathbb{Z}/k\mathbb{Z}$ -manifold. By [13], $(Q, \eta_{\mathbb{C}})$ defines a cycle in $K^*(Y; \mathbb{Q}/\mathbb{Z}) := \lim K^*(Y; \mathbb{Z}/k\mathbb{Z})$. The reader should note that, when $Y = pt$, the Freed-Melrose index (see [14] and [17]) of this cycle produces the required element of \mathbb{Q}/\mathbb{Z} .

The construction of the element of $K^*(Y; \mathbb{R}) \cong K^*(Y; N)$ (associated to (W, ξ)) proceeds as follows. Form a cycle in $K^*(Y; N)$ via

$$(5) \quad (kW, \frac{1}{k}\xi_N)^{N_W} \cup_{k(\partial W)^N} (-Q, \frac{1}{k}\phi_*(\eta_{\mathbb{C}}))$$

Notice that the choice of K -theory class in this cycle is not unique; it depends on a choice of clutching function, which we denote by u . The well-definedness of this construction will be discussed shortly. Again, when $Y = pt$, one takes the \mathbb{R} -valued index of this cycle to produce the required element in \mathbb{R} .

Let $\text{ind}_{\mathbb{R}/\mathbb{Z}}$ denote the map produced by these two constructions. Our first goal is to prove that this map is well-defined. The reader should note that the map is

not well-defined as a map to $K^*(Y; \mathbb{Q}/\mathbb{Z}) \oplus K^*(Y; \mathbb{R})$. The point being that we have made a number of choices in the construction of the image of (Z, ξ) under $\text{ind}_{\mathbb{R}/\mathbb{Z}}$.

To summarize, these choices are as follows:

- (1) The $k \in \mathbb{N}$;
- (2) The normal bundle N_W over W ;
- (3) The bounding cycle $(Q, \eta_{\mathbb{C}})$;
- (4) The choice of clutching function u .

As above, let N denote the restriction of N_W to ∂W . Using the obvious notation, let k' , N'_W , (Q', η') , and u' be different choices of the previously listed data. By assumption,

$$\begin{aligned} k(\partial W, \xi_{\mathbb{C}})^N &= \partial(Q, \eta_{\mathbb{C}}) \\ k'(\partial W, \xi_{\mathbb{C}})^{N'} &= \partial(Q', \eta'_{\mathbb{C}}) \end{aligned}$$

To begin, the nature of the inductive limits used to define K -theory with coefficients in \mathbb{Q} and \mathbb{Q}/\mathbb{Z} implies that we can assume $k = k'$; in other words, one can replace k and k' with $k \cdot k'$.

By Lemma 4.15, there exists trivial bundles, ϵ and ϵ' over W , such that

$$N \oplus \epsilon \cong N' \oplus \epsilon$$

Lemma 4.16 implies that

$$((W, \xi)^N)^{p_1^*(\epsilon)} \sim_{\text{bor}} ((W, \xi)^{N'})^{p_2^*(\epsilon')}$$

Let $((Z, \tilde{W} \dot{\cup} \tilde{W}'), \tilde{\xi})$ be a fixed choice of such a bordism. Then $(\partial Z - \text{int}(\tilde{W} \dot{\cup} \tilde{W}'), \tilde{\xi}_{\mathbb{C}})$ defines a bordism in $K_*(pt)$ between $((\partial W, \xi_{\mathbb{C}})^N)^{p_1^*(\epsilon)}$ and $((\partial W, \xi_{\mathbb{C}})^{N'})^{p_2^*(\epsilon')}$.

As such, we can form an element in $K^*(Y; \mathbb{Q})$. Let

$$(Q, \eta_{\mathbb{C}})^{\epsilon} \cup k(\partial Z - \text{int}(\tilde{W} \dot{\cup} \tilde{W}'), \tilde{\xi}_{\mathbb{C}}) \cup (Q', \eta'_{\mathbb{C}})^{\epsilon'}$$

be the Baum-Douglas cycle formed by gluing along $\partial Z = \partial W \dot{\cup} -\partial W$ and using the clutching function $(u')^{-1} \circ u$. Then consider this cycle as an element in $K^*(Y; \mathbb{Q})$ using the inductive limit structure on \mathbb{Q} . To be more precise, this cycle is considered as a element in $K^*(Y; \mathbb{Q})$ using the following commutative diagram:

$$\begin{array}{ccc} K^*(Y) & \longrightarrow & K^*(Y; \mathbb{Q}) \\ \downarrow & & \downarrow \\ K^*(Y; \mathbb{Z}/k\mathbb{Z}) & \longrightarrow & K^*(Y; \mathbb{Q}/\mathbb{Z}) \end{array}$$

Using the $\mathbb{Z}/k\mathbb{Z}$ -bordisms and vector bundle modification relations (see [13]), we have

$$\begin{aligned} (Q, \eta_{\mathbb{C}}) &\sim (Q, \eta_{\mathbb{C}})^{\epsilon_Q} \\ &\sim (Q, \eta_{\mathbb{C}})^{\epsilon_Q} \dot{\cup} (-Q', \eta_{\mathbb{C}})^{\epsilon_{Q'}} \dot{\cup} (Q', \eta_{\mathbb{C}})^{\epsilon_{Q'}} \\ &\sim \delta_{\mathbb{Z}/k\mathbb{Z}} \left((Q, \eta_{\mathbb{C}})^{\epsilon} \cup k(\partial Z - \text{int}(\tilde{W} \dot{\cup} \tilde{W}'), \tilde{\xi}_{\mathbb{C}}) \cup (Q', \eta'_{\mathbb{C}})^{\epsilon'} \right) \dot{\cup} (Q', \eta_{\mathbb{C}})^{\epsilon_{Q'}} \\ &\sim \delta_{\mathbb{Z}/k\mathbb{Z}} \left((Q, \eta_{\mathbb{C}})^{\epsilon} \cup k(\partial Z - \text{int}(\tilde{W} \dot{\cup} \tilde{W}'), \tilde{\xi}_{\mathbb{C}}) \cup (Q', \eta'_{\mathbb{C}})^{\epsilon'} \right) \dot{\cup} (Q', \eta_{\mathbb{C}}) \end{aligned}$$

This bordism and the commutative diagram discussed in the previous paragraph imply that the construction of the element of $K^*(Y; \mathbb{Q}/\mathbb{Z})$ is unique up to the

image of an element in the image of q . The reader should note the specific element of $K^*(Y; \mathbb{Q})$ is given by the cycle

$$(Q, \eta_{\mathbb{C}})^{\epsilon} \cup k(\partial Z - \text{int}(\tilde{W} \dot{\cup} \tilde{W}'), \tilde{\xi}_{\mathbb{C}}) \cup (Q', \eta'_{\mathbb{C}})^{\epsilon'}$$

To complete the proof of well-definedness, we must show that the cycle $(kW, \frac{1}{k}\xi_N) \cup_{k\partial W} (-Q, \frac{1}{k}\phi_*(\eta_{\mathbb{C}}))$ is equivalent to

$$(kW, \frac{1}{k}\xi_N) \cup_{k\partial W} (-Q', \frac{1}{k}\phi_*(\eta'_{\mathbb{C}})) \dot{\cup} (Q, \eta_{\mathbb{C}})^{\epsilon} \cup k(\partial Z - \text{int}(\tilde{W} \dot{\cup} \tilde{W}'), \tilde{\xi}_{\mathbb{C}}) \cup (Q', \eta'_{\mathbb{C}})^{\epsilon'}$$

This follows from the bordism relation in $K^*(Y; N)$ and the existence of the bordism $((Z, \tilde{W} \dot{\cup} \tilde{W}'), \frac{1}{k} \cdot \phi_*(\xi))$. Thus $\text{ind}_{\mathbb{R}/\mathbb{Z}}$ is well-defined.

Proposition 6.1. *The map $\text{ind}_{\mathbb{R}/\mathbb{Z}} : K^0(Y; \phi) \rightarrow \text{coker}(q)$ is an isomorphism.*

Proof. This follows from the Five Lemma and the commutative diagram:

$$\begin{array}{ccccccccc} \longrightarrow & K^1(Y; \mathbb{R}/\mathbb{Z}) & \longrightarrow & K^0(Y) & \longrightarrow & K^0(Y; \mathbb{R}) & \longrightarrow & K^0(Y; \mathbb{R}/\mathbb{Z}) & \longrightarrow & K^1(Y) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & \text{coker}(q^1) & \longrightarrow & K^0(Y) & \longrightarrow & K^0(Y; \mathbb{R}) & \longrightarrow & \text{coker}(q^0) & \longrightarrow & K^1(Y) & \longrightarrow \end{array}$$

□

Returning to the case when Y is a point, the composition of $\text{ind}_{\mathbb{R}/\mathbb{Z}}$ with the map $(W, \xi, f) \rightarrow (W, \xi)$ gives an \mathbb{R}/\mathbb{Z} -valued index map on $K_0(X; \mathbb{R}/\mathbb{Z})$.

6.2. Index pairings and slant products. A review of K -theory with coefficients in \mathbb{R}/\mathbb{Z} is required before discussing the various index pairings related to our theory. In [4], a model for $K^*(X; \mathbb{R}/\mathbb{Z})$ is constructed using ideas of Karoubi. Cocycles in $K^1(X; \mathbb{R}/\mathbb{Z})$ are given by triples, (V_1, V_2, φ) , where V_i are vector bundle over X and $\alpha : V_1 \otimes_{\phi} N \rightarrow V_2 \otimes_{\phi} N$ is an isomorphism of N -module bundles. In other words, the K -theory of X with coefficients in \mathbb{R}/\mathbb{Z} is given by $K^*(X; \phi)$ (as defined in Section 2.1).

Proposition 6.2. *Let X be a finite CW-complex and Y be a compact Hausdorff space. Then, we have well-defined slant product:*

$$K^q(Y \times X; \mathbb{R}/\mathbb{Z}) \times K_p(X) \rightarrow K^{p+q}(Y; \mathbb{R}/\mathbb{Z})$$

In the case $Y = pt$, the slant product reduces to an index pairing; an element of \mathbb{R}/\mathbb{Z} is obtained by taking the index of cycle in $K_0(pt; \mathbb{R}/\mathbb{Z})$.

We give a detailed treatment for the odd slant products; the even products are obtained using a similar construction.

For the slant product $K^1(Y \times X; \mathbb{R}/\mathbb{Z}) \times K_1(X)$, fix a cocycle, $(V_1, V_2, \varphi) \in K^1(Y \times X; \mathbb{R}/\mathbb{Z})$, and a cycle, $(M, E, f) \in K_1(X)$. Define the slant product at the level of cycles to be

$$(M \times [0, 1], (\pi_M^*(E) \otimes (f^*(V_1) \otimes_{\phi} N), F, \alpha))$$

where

- (1) $\pi_M : M \times [0, 1] \rightarrow M$ is the projection map;
- (2) F is the vector bundle defined by $f^*(V_1)$ on $Y \times M \times \{0\}$ and $f^*(V_2)$ on $Y \times M \times \{1\}$;

- (3) α is the identity on the fibers at $Y \times M \times \{0\}$ and φ on the fibers at $Y \times M \times \{1\}$;

To show that this slant product is well-defined, we must show that it is invariant under the relations in both K -theory and K -homology. Starting with the relations in K -theory, suppose that (V_1, V_2, φ) is an elementary cocycle and that φ_t is a fixed homotopy from $Id_{\phi_*(V_1)}$ to φ . Let D denote the closed unit disk. Then, the bordism

$$((M \times D, M \times [0, 1]), (\pi_{[0,1]}^*(f^*(V_1) \otimes E), \phi_*(\pi_D^*(f^*(V_1) \otimes E)), h_t))$$

has boundary given by the slant product of (V_1, V_2, φ) with (M, E, f) ; hence the slant product vanishes for elementary cocycles. A similar construction implies that the construction respects isomorphism of cocycles.

Next, consider the relations in the K -homology group. The slant product clearly respects the disjoint union operation and relation. For the bordism relation, suppose that (W, \bar{E}, g) is a cycle with boundary, (M, E, f) . By straightening the angle, we have a smooth compact $spin^c$ -manifold (with boundary), Z , which is homeomorphic (via h) to $W \times [0, 1]$. Moreover, ∂Z has a regular domain given by $M \times [0, 1]$ and

$$\partial Z - (M \times (0, 1)) = W \dot{\cup} - W$$

Thus

$$\begin{aligned} \langle (V_1, V_2, \varphi), (M, E, f) \rangle &= (M \times [0, 1], (\pi_M^*(E) \otimes (f^*(V_1) \otimes_\phi N), F, \alpha)) \\ &= \partial(Z, M \times [0, 1], ((\pi_W \circ h)^*(\bar{E}) \otimes g^*(V_1) \otimes_\phi N, \bar{F}, \bar{\alpha})) \end{aligned}$$

where

- (1) $\pi_W : W \times [0, 1] \rightarrow W$ is the projection map;
- (2) F is the vector bundle defined by $(\bar{E} \otimes f^*(V_1))$ on $W \times \{0\}$ and $(\bar{E} \otimes f^*(V_2))$ on $W \times \{1\}$;
- (3) $\bar{\alpha}$ is the identity on the fibers at $W \times \{0\}$ and φ on the fibers at $W \times \{1\}$;

Finally, for vector bundle modification, let (V_1, V_2, φ) be a cocycle in $K^1(Y \times X; \mathbb{R}/\mathbb{Z})$, (M, E, f) be a Baum-Douglas cycle and V be an even rank $spin^c$ -vector bundle over M . Then, the well-definedness of the slant product follows since

$$(M \times [0, 1], (\pi_M^*(E) \otimes (V_1 \otimes_\phi N), F, \alpha)) \sim (M \times [0, 1], (\pi_M^*(E) \otimes (V_1 \otimes_\phi N), F, \alpha))^{\pi^*(V)}$$

This completes the proof that the slant product $K^1(Y \times X; \mathbb{R}/\mathbb{Z}) \times K_1(X) \rightarrow K_0(Y; \mathbb{R}/\mathbb{Z})$ is well-defined.

For the slant product $K^1(Y \times X) \times K_1(X; \mathbb{R}/\mathbb{Z})$, fix a unitary, $u : Y \times X \rightarrow \mathcal{U}(n)$, and a cycle, $(Q, (E_{B_2}, F_{B_1}, \alpha), f)$ and define their slant product to be the cycle:

$$(W \times S^1, (\pi_W^*(E_{B_2}) \otimes (V_u \otimes_\phi N), \pi_W^*(F_{B_1}) \otimes V_u, \alpha \otimes id))$$

where

- (1) $\pi_W : W \times S^1$ is the projection onto W ;
- (2) V_u is the vector bundle obtained by clutching the trivial bundle $Y \times W \times [0, 1] \times \mathbb{C}^n$ using the automorphism associated to the function $u \circ (id_Y \times f) : Y \times W \rightarrow \mathcal{U}(n)$.

The proof that this slant product is well-defined is as follows. Beginning with the relations in K -theory, suppose u_t is a continuous path of unitaries. For any t , $(\pi_W^*(E_{B_2}) \otimes V_{u_t}, \pi^*(F_{B_1}) \otimes V_{u_t}, \alpha \otimes id)$ determine the same class in $K^0(Y \times W, Y \times \partial W; \phi)$. Since the slant product depends only on the K -theory class in

$K^0(Y \times W, Y \times \partial W; \phi)$, hence it is independent of the particular unitary representative in a class in $K^1(Y \times X)$.

The relations in K -homology are considered next. The proof for the disjoint union/direct sum relation is trivial. For bordism, suppose that $(W, (E_N, F_{\mathbb{C}}, \alpha), f)$ is the boundary of the bordism $((Q, W), (E'_N, F'_{\mathbb{C}}, \alpha'), g)$. Then form the bordism (in $K_0(Y; \mathbb{R}/\mathbb{Z})$)

$$(Q \times S^1, W \times S^1), (\pi_Q^*(E_{B_2}) \otimes (V_u \otimes_{\phi} N), \pi_{\partial Q - \text{int}(W)}^*(F_{B_1}) \otimes V_u, \alpha' \otimes id))$$

where the notation is as in the definition of the slant product (e.g., $\pi_Q : Q \times S^1 \rightarrow Q$ is the projection map).

Finally, for the vector bundle modification relation, let V be a $spin^c$ vector bundle over W of even rank. Then the slant product of $[u]$ with $(W, (E_N, F_{\mathbb{C}}, \alpha), f)^V$ is given by

$$(W \times S^1, (\pi_W^*(E_{B_2}) \otimes (V_u \otimes_{\phi} N), \pi_W^*(F_{B_1}) \otimes V_u, \alpha \otimes id))^{p^*(V)}$$

where $p : W \times S^1 \rightarrow W$. This completes the proof that the slant product, $K^1(Y \times X) \times K_1(X; \mathbb{R}/\mathbb{Z}) \rightarrow K_0(Y; \mathbb{R}/\mathbb{Z})$ is well-defined.

6.3. Relationship with Lott's pairing. In [22], Lott discusses a model for K -theory with coefficients in \mathbb{R}/\mathbb{Z} for smooth manifolds using connections and differential forms. Again, this construction is based on work of Karoubi. In particular, Lott constructs a pairing between K -theory with coefficients in \mathbb{R}/\mathbb{Z} and K -homology which is given by the relative η -invariant. The main goal of this section is a proof that the pairing defined in the previous section is equal to Lott's pairing.

In this section, we must restrict to the case when X is a smooth manifold. Let $K_{Lott}^*(X; \mathbb{R}/\mathbb{Z})$ denote the realization of K -theory with coefficients in \mathbb{R}/\mathbb{Z} discussed in [22]. We will only discuss the odd pairing in detail. A cocycle in $K_{Lott}^1(X; \mathbb{R}/\mathbb{Z})$ is given by $((V_1, \nabla_1), (V_2, \nabla_2), \omega)$ where V_1 and V_2 are complex Hermitian vector bundles, ∇_1 and ∇_2 are Hermitian connections on V_1 and V_2 respectively, and $\omega \in \Omega^{odd}(M)/im(d)$ such that $d\omega = ch(\nabla_1) - ch(\nabla_2)$.

Recall (see [4]) that the isomorphism from $K_{Basu}^1(X; \mathbb{R}/\mathbb{Z})$ to $K_{Lott}^1(X; \mathbb{R}/\mathbb{Z})$ is defined at the level of cocycles via

$$(V_1, V_2, \varphi) \mapsto ((V_1, \nabla_1), (V_2, \nabla_2), CS_N(\tilde{\nabla}_1, \varphi^*(\tilde{\nabla}_2)))$$

where

- (1) ∇_1 and ∇_2 are connections on V_1 and V_2 respectively;
- (2) For $i = 1$ or 2 , $\tilde{\nabla}_i := \nabla_i \otimes I + I \otimes d$ is a N -bundle connection on $V_i \otimes_{\phi} N$;
- (3) $CS_N(\nabla_1, \varphi^*(\nabla_2))$ is the Chern-Simon form associated to the N -bundle connections $\tilde{\nabla}_1$ and $\varphi^*(\tilde{\nabla}_2)$.

Further details on these cocycles and $K_{Lott}^*(X; \mathbb{R}/\mathbb{Z})$ can be found in [22]. The reader can find more details on the η -invariant in [1, 2, 3].

The pairing $K_{Lott}^1(X; \mathbb{R}/\mathbb{Z}) \times K_1(X) \rightarrow \mathbb{R}/\mathbb{Z}$ is given at the level of cocycle and cycle via

$$\begin{aligned} \langle ((V_1, \nabla_1), (V_2, \nabla_2), \omega), (M, E, f) \rangle &:= \bar{\eta}(D_{f^*(\nabla_1)}) - \bar{\eta}(D_{f^*(\nabla_2)}) \\ &\quad - \int_M \text{Todd}(M) \wedge ch(E) \wedge f^*(\omega) \mod \mathbb{Z} \end{aligned}$$

where

- (1) f is assumed to be a smooth function;

- (2) $D_{f^*(\nabla_i)}$ is the Dirac operator on M twisted by $E \otimes f^*(V_i)$;
- (3) $\eta(D_{f^*(\nabla_i)})$ is the η -invariant associated to $D_{f^*(\nabla_i)}$;
- (4) $\bar{\eta}(D_{f^*(\nabla_i)}) = \frac{\eta(D_{f^*(\nabla_i)}) + \dim(\ker(D_{f^*(\nabla_i)}))}{2} \bmod \mathbb{Z}$.

Theorem 6.3. *Let X be a smooth compact manifold. Then, the following diagram commutes:*

$$\begin{array}{ccc} K_{Basu}^1(X; \mathbb{R}/\mathbb{Z}) \times K_1(X) & \xrightarrow{(\Phi \times Id)} & K_{Lott}^1(X; \mathbb{R}/\mathbb{Z}) \times K_1(X) \\ \downarrow & & \downarrow \\ K_0(pt; \mathbb{R}/\mathbb{Z}) & \xrightarrow{\text{ind}_{\mathbb{R}/\mathbb{Z}}} & \mathbb{R}/\mathbb{Z} \end{array}$$

where

- (1) The vertical maps are the pairings;
- (2) $\Phi : K_{Basu}^1(X; \mathbb{R}/\mathbb{Z}) \rightarrow K_{Lott}^1(X; \mathbb{R}/\mathbb{Z})$ is the isomorphism constructed in [4] (see the introduction of this subsection);
- (3) $\text{ind}_{\mathbb{R}/\mathbb{Z}}$ is the index map defined in Section 6.

In other words, the pairing defined in [22] is equal to the pairing defined in Section 6.2.

Proof. Let (V_1, V_2, φ) be a cocycle in $K_{Basu}^1(X; \mathbb{R}/\mathbb{Z})$, (M, E, f) be a Baum-Douglas cycle in $K_1(X)$, and $((V_1, \nabla_1), (V_2, \nabla_2), CS_N(\tilde{\nabla}_1, \varphi^*(\tilde{\nabla}_2)))$ denote the image of (V_1, V_2, φ) under the isomorphism from $K_{Basu}^1(X; \mathbb{R}/\mathbb{Z})$ to $K_{Lott}^1(X; \mathbb{R}/\mathbb{Z})$. To streamline the proof, we introduce some notation. Let η_i denote $\bar{\eta}(D_{f^*(\nabla_i)})$ and \mathfrak{F} denote the cycle $(M \times [0, 1], (\pi^*(E \otimes f^*(V_1)) \otimes_\phi N, F, \alpha)$; the reader should recall the notation of Section 6.2.

In this notation, the proof amounts to showing the following equality:

$$\text{ind}_{\mathbb{R}/\mathbb{Z}}(\mathfrak{F}) = \eta_1 - \eta_2 - \int_M \text{Todd}(M) \wedge ch(E) \wedge f^*(CS_N(\tilde{\nabla}_1, \varphi^*(\tilde{\nabla}_2)))$$

The computation of these indices requires us to fix quite a bit of data. We complete the proof assuming that (M, E, f) satisfies the property that $(M \times \{0\} \dot{\cup} M \times \{1\}, E \otimes f^*(V_1) \dot{\cup} E \otimes f^*(V_2))$ is a boundary as a Baum-Douglas cycle over a point. The general case can be obtained using the model using K -theory classes and the existence of a normal bordism to a cycle with this property. Let

- (1) (Q, G) be a particular choice of bounding cycle for $(M \times \{0\} \dot{\cup} M \times \{1\}, E \otimes f^*(V_1) \dot{\cup} E \otimes f^*(V_2))$.
- (2) Z be the compact $spin^c$ -manifold without boundary $Q \cup (M \times [0, 1])$ and (Z, F_N) be the cycle in $K_*(X; N)$ formed (via clutching the bundles using α) from $(M \times [0, 1], \pi^*(E) \otimes (f^*(V_1)))$ and (Q, G) ;

In addition, the Dirac operators associated to the various manifolds twisted by the appropriate vector or von Neumann bundle will be denoted using subscript notation (e.g., the Dirac operator on Q will be denote by D_Q). Then

$$\text{ind}_{\mathbb{R}/\mathbb{Z}}(\mathfrak{F}) = (0, \text{ind}_N(D_{Q \cup M \times [0, 1]})) \in \text{coker}(q)$$

where $q : \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \oplus \mathbb{R}$ is the natural map (see Section 6.1). Further geometric structures on the manifolds and bundles involved are required. Without loss of generality (see [22, Proposition 3]), assume that f is the identity map and E is a trivial line bundle (this simplifies notation).

- (1) Let g be a metric on Z which is a product in a neighbourhood of $M \times [0, 1] \subseteq Z$;
- (2) Fix a connection, ∇_F on F which is compatible with g ;
- (3) Fix an N -connection, ∇_N on F_N which is compatible with g and is equal to $\nabla_F \otimes I + I \otimes d$ on fibers over $Q \subseteq Z$. Moreover, choose ∇_N so that it is of the form $\tilde{\nabla}_1 t + (1-t)\varphi^*(\tilde{\nabla})_2$ on $M \times [0, 1]$ where $\tilde{\nabla}_i := \nabla_i \otimes I + I \otimes d$;

Using the specific geometric data above, we have

$$\begin{aligned}
\text{ind}_N(D_{Q \cup M \times [0,1]}) &= \int_{Q \cup M \times [0,1]} \text{Todd}(Z) ch_N(\nabla_N) \in \mathbb{R} \\
&= \int_Q \text{Todd}(Q) ch_N(\nabla_N|_Q) \\
&\quad + \int_{M \times [0,1]} \text{Todd}(M \times [0,1]) ch_N(\nabla_N|_{M \times [0,1]})
\end{aligned}$$

The isomorphism from $\text{coker}(q)$ to \mathbb{R}/\mathbb{Z} is given by taking the difference of the \mathbb{Q}/\mathbb{Z} -entry and \mathbb{R} -entry mod \mathbb{Z} . Thus,

$$\text{ind}_{\mathbb{R}/\mathbb{Z}}(\mathfrak{F}) = -\text{ind}_N(D) \bmod \mathbb{Z}$$

The fact that the formula for $\text{ind}_N(D_{Q \cup M \times [0,1]})$ is local and the specific nature of the connections used lead to

$$\text{ind}_{\mathbb{R}/\mathbb{Z}}(\mathfrak{F}) = \eta_1 - \eta_2 - \int_{M \times [0,1]} \text{Todd}(M \times [0,1]) ch_N(\nabla_N|_{M \times [0,1]}) \bmod \mathbb{Z}$$

The fact that $\nabla_N|_{M \times [0,1]} = t\tilde{\nabla}_1 + (1-t)\varphi^*(\tilde{\nabla}_2)$, the definition of the Chern-Simon form and the fact that the other differential forms involved are pullbacks further reduces this expression to

$$\text{ind}_{\mathbb{R}/\mathbb{Z}}(\mathfrak{F}) = \eta_1 - \eta_2 - \int_M \text{Todd}(M) CS_N(\tilde{\nabla}_1, \varphi^*(\nabla_2)) \bmod \mathbb{Z}$$

This completes the proof. \square

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